On a Two-Variable Zeta Function for Number Fields

Jeffrey C. Lagarias *
Eric Rains
AT&T Labs - Research
Florham Park, NJ 07932

(July 6, 2002 version)

Abstract

This paper studies a two-variable zeta function $Z_K(w,s)$ attached to an algebraic number field K, introduced by van der Geer and Schoof [11], which is based on an analogue of the Riemann-Roch theorem for number fields using Arakelov divisors. When $w=1$ this function becomes the completed Dedekind zeta function $\hat{\zeta}_K(s)$ of the field K. The function is an meromorphic function of two complex variables with polar divisor $s(w-s)$, and it satisfies the functional equation $Z_K(w,s)=Z_K(w,w-s)$. We consider the special case $K=\mathbb{Q}$, where for $w=1$ this function is $\hat{\zeta}(s)=\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$. The function $\xi_{\mathbb{Q}}(w,s):=\frac{s(s-w)}{2w}Z_{\mathbb{Q}}(w,s)$ is shown to be an entire function on $\mathbb{C}^2$, to satisfy the functional equation $\xi_{\mathbb{Q}}(w,s)=\xi_{\mathbb{Q}}(w,w-s)$, and to have $\xi_{\mathbb{Q}}(0,s)=-\frac{s^2}{8}(1-2^{1+\frac{s}{2}})(1-2^{1-\frac{s}{2}})\hat{\zeta}(\frac{s}{2})\hat{\zeta}(\frac{-s}{2})$. We study the location of the zeros of $Z_{\mathbb{Q}}(w,s)$ for various real values of $w=u$. For fixed $u\geq 0$ the zeros are confined to a vertical strip of width at most $u+16$ and the number of zeros $N_u(T)$ to height T has similar asymptotics to the Riemann zeta function. For fixed $u<0$ these functions are strictly positive on the "critical line" $\Re(s)=\frac{u}{2}$. This phenomenon is associated to a positive convolution semigroup with parameter $u\in\mathbb{R}_{>0}$, which is a semigroup of infinitely divisible probability distributions. having densities $P_u(x)dx$ for real x, where $P_u(x)=\frac{1}{2\pi}\theta(1)^u Z_{\mathbb{Q}}(-u,-\frac{u}{2}+ix)$, and $\theta(1)=\pi^{1/4}/\Gamma(3/4)$.

AMS Subject Classification (2000): 11M41 (Primary) 14G40, 60E07 (Secondary)

Keywords: Arakelov divisors, functional equation, zeta functions

1. Introduction

Recently van der Geer and Schoof [11, Prop. 1] formulated an "exact" analogue of the Riemann-Roch theorem valid for an algebraic number field K, based on Arakelov divisors. They used this result to formally express the completed zeta function $\hat{\zeta}_K(s)$ of K as an integral over the Arakelov divisor class group $Pic(K)$ of K. They introduced a two-variable zeta function attached to a number field K, also given as an integral over the Arakelov class group, which we call either the *Arakelov zeta function* or the *two-variable zeta function*. This zeta function was modelled after a two-variable zeta function attached to a function field over a finite field, introduced in 1996 by Pellikaan [18]. For convenience we review the Arakelov divisor interpretation of the two-variable zeta function and the Riemann-Roch theorem for number fields in an appendix.

*Work done in part during a visit to the Institute for Advanced Study.

In this paper we study in detail the two-variable zeta function attached to the rational field $K = \mathbb{Q}$. Then in the final section we consider two-variable zeta functions for general algebraic number fields K. The results are derived starting from an integral representation of this function, and if one takes it as given, then the paper is independent of the Arakelov divisor interpretation. The Arakelov class group of $\mathbb{Q}$ can be identified with the positive real line (with multiplication as the group operation) and van der Geer and Schoof's integral becomes, formally,

$$Z_{\mathbb{Q}}(s) \cong \int_0^\infty \theta(t^2)^s \theta(\frac{1}{t^2})^{1-s} \frac{dt}{t}, \tag{1.1}$$

in which

$$\theta(t) := \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t} \tag{1.2}$$

is the theta function $\theta(t) = \vartheta_3(0, e^{\pi t})$, where

$$\vartheta_3(z, q) := \sum_{n \in \mathbb{Z}} e^{2\pi i n z} q^{n^2} = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos 2nz,$$

is a Jacobi theta function. The $\cong$ used in (1.1) reflects the fact that the integral on its right side converges *nowhere*; a regularization is needed to assign it a meaning. Such a regularization can be obtained using the Arakelov two-variable zeta function $Z_{\mathbb{Q}}(w, s)$ attached to $\mathbb{Q}$, which we define to be

$$Z_{\mathbb{Q}}(w, s) := \int_0^\infty \theta(t^2)^s \theta(\frac{1}{t^2})^{w-s} \frac{dt}{t} \ . \tag{1.3}$$

Our definition here differs from the one in van der Geer and Schoof [11] by a linear change of variable, setting their second variable $t = w - s$. The integral on the right side of (1.3) has a region of absolute convergence in $\mathbb{C}^2$, which is the open cone

$$\mathcal{C} := \{(w, s) : \Re(w) < \Re(s) < 0\}. \tag{1.4}$$

The function $Z_{\mathbb{Q}}(w, s)$ meromorphically continues from the cone $\mathcal{C}$ to all of $\mathbb{C}^2$, with polar divisor consisting of the (complex) hyperplanes $\{s = w\} \cup \{s = 0\}$, a set of real-codimension two, see §2. On restricting $Z_{\mathbb{Q}}(w, s)$ to the line $w = 1$, the resulting function is the *completed Riemann zeta function* $\hat{\zeta}(s)$, which is

$$\hat{\zeta}(s) := \pi^{-s/2} \Gamma(s/2) \zeta(s).$$

Thus the two-variable zeta function $Z_{\mathbb{Q}}(w, s)$ defined via (1.3) provides a method to regularize the integral (1.1), and the same can be done for arbitrary number fields.

We are motivated by several questions about this function.

(1) What are the properties of the function as a meromorphic function of two complex variables? In particular, determine information about its zero divisor.

(2) What is the meaning of the additional variable w and what arithmetic information does it encode?

(3) What properties of this two-variable zeta function reflect Arakelov geometry?

(4) Is there any connection between zeta functions encoding information based on Arakelov geometry and zeta functions coming from automorphic representations and the Langlands program?

This paper mainly addresses question (1), obtaining information on the zero set of the Arakelov zeta function. Concerning question (2), we observe in §2 that the function $Z_{\mathbb{Q}}(w,s)$ is representable by the integral

$$Z_{\mathbb{Q}}(w,s) = \frac{1}{2}\int_0^{\infty} \theta(t)^w t^{s/2}\frac{dt}{t}, \tag{1.5}$$

which expresses it as a Mellin transform of $2\theta(t)^w$. The function $\theta(t)^w$ is a modular form of weight $\frac{w}{2}$ (with multiplier system) on a congruence subgroup of the modular group, and the complex variable $w/2$ parametrizes the *weight* of this modular form. The arithmetic information it encodes includes the invariants $\eta(\mathbb{Q})$ and $g(\mathbb{Q})$ introduced in van der Geer and Schoof [11], defined in the appendix. Concerning question (3), we observe that there is an extra structure associated to $Z_{\mathbb{Q}}(w,s)$, which is a holomorphic convolution semigroup of complex-valued measures on lines $L(t) = \{(u, u+it) : -\infty < t < \infty\}$ in the real-codimension one cone

$$\mathcal{C}^- := \{(w,s) : w = u \in \mathbb{R} \;\text{ and }\; u < \Re(s) < 0\}$$

see §7.2. This cone is contained in the region of absolute convergence $\mathcal{C}$ of the integral representation (1.3). Of particular interest for $Z_{\mathbb{Q}}(w,s)$ is the real-codimension two subcone

$$\mathcal{C}_{crit} = \{(w,s) : w = u \in \mathbb{R} \;\text{ and } \Re(s) = \frac{u}{2} < 0\},$$

which generalizes the "critical line" of the zeta function, and on which the measures are real-valued. Perhaps this semigroup structure is associated in some way with Arakelov geometry, since various constants associated with the semigroup on the subcone $\mathcal{C}_{crit}$ have arithmetic interpretations in the framework of van der Geer and Schoof, see §7. Concerning question (4), the subject of Arakelov geometry was developed in part to answer Diophantine questions and has a completely different origin from automorphic representations. Any connection between these two subjects could potentially be of great interest. However we do not find any obvious connection, and note only that the w variable interpolates between modular forms of different weights, and when w is a positive even integer these are holomorphic modular forms of the type appearing in automorphic representations. In general these forms are not eigenforms for Hecke operators, and in §3.4 we show these forms have associated Euler products exactly when $w = 0, 1, 2, 4$ and 8.

Besides giving information on questions (1)-(4) above, the analysis of this paper may be useful for other purposes. This function provides an interesting example of an entire function in two complex variables of finite order, see Ronkin [24] and Stoll [25]. The information about the zero locus of $Z_{\mathbb{Q}}(w,s)$ that we obtain mainly concerns the region where where the variable w is real; these are Mellin transforms of modular forms of real weight, which have been extensively studied. The movements of zeros in the s-plane as the (real) parameter w is varied may be compared with movement of zeros under milder deformations such as those in linear combinations of L-functions, see Bombieri and Hejhal [5] and Hejhal [12].

The function $Z_{\mathbb{Q}}(w,s)$ shares many properties of the Riemann zeta function. It satisfies the functional equation

$$Z_{\mathbb{Q}}(w,s) = Z_{\mathbb{Q}}(w, w-s)\ . \tag{1.6}$$

When $w = u$ is real then $Z_{\mathbb{Q}}(u,s)$ retains several familiar symmetries of the Riemann zeta function: it is real on the real axis $\Im(s) = 0$, and it is real on the "critical line" $\mathrm{Re}(s) = \frac{u}{2}$, which is the line of symmetry of the functional equation. Thus for fixed real u, the zeros of $Z_{\mathbb{Q}}(u,s)$ which do not lie on the critical line or the real axis must occur in sets of four: s, $u-s$, $\bar{s}$, $u-\bar{s}$. This extra symmetry can be used to extract information about the components of the zero locus, see Lemma 8.1. On the other hand, for most real u the function $Z_{\mathbb{Q}}(u,s)$ fails to satisfy a Riemann hypothesis, as we describe below. The Riemann zeta function appears when $w = 1$ and it is interesting to note that it also appears in terms of data at $w = 0$, given in the following result, which is proved in §5.

Theorem 1.1. *The function*

$$\xi_{\mathbb{Q}}(w,s) = \frac{s(s-w)}{2w} Z_{\mathbb{Q}}(w,s)$$

is an entire function in two complex variables. At $w = 0$ it is

$$\xi_{\mathbb{Q}}(0,s) = -\frac{s^2}{8}(1-2^{1+\frac{s}{2}})(1-2^{1-\frac{s}{2}})\hat{\zeta}(\frac{s}{2})\hat{\zeta}(\frac{-s}{2}), \tag{1.7}$$

where $\hat{\zeta}(s)$ is the completed Riemann zeta function. In particular, for all real t,

$$\xi_{\mathbb{Q}}(0,it) = \frac{t^2}{8}|1-2^{1+\frac{it}{2}}|^2|\hat{\zeta}(\frac{it}{2})|^2 \tag{1.8}$$

is strictly positive, with $\xi_{\mathbb{Q}}(0,0) = 1/2$.

The Jacobi triple product formula plays an essential role in our derivation of the formula (1.7). Note that the function $\xi_{\mathbb{Q}}(1,s)$ coincides with the Riemann ξ-function, and the functional equation $\xi_{\mathbb{Q}}(w,s) = \xi_{\mathbb{Q}}(w,w-s)$ is inherited from $Z_{\mathbb{Q}}(w,s)$.

The most striking result of this paper appears in §7, and concerns for negative real $w = -u$ $(u > 0)$ the behavior of the function $Z_{\mathbb{Q}}(w,s)$ on the "critical line" $\Re(s) = -\frac{u}{2}$. This result makes a connection with probability theory, involving infinitely divisible distributions.

Theorem 1.2. *For negative real w, with $w = -u$ $(u > 0)$ the function $Z_{\mathbb{Q}}\left(-u, -\frac{u}{2} + it\right)$ is given as the Fourier transform*

$$Z_{\mathbb{Q}}\left(-u, -\frac{u}{2} + it\right) = \theta(1)^{-|u|} \int_{-\infty}^{\infty} f(r)^u e^{-irt} dr \tag{1.9}$$

in which

$$f(r) = \theta(1)\frac{e^{r/2}}{\theta(e^{-2r})} = \frac{\theta(1)}{\sqrt{\theta(e^{-2r})\theta(e^{2r})}} = \theta(1)\frac{e^{-r/2}}{\theta(e^{2r})}$$

and

$$\theta(e^{-2r}) = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2r}}.$$

The function $f(r)$ is the characteristic function of an infinitely divisible probability measure with finite second moment, whose associated Khintchine canonical measure $\frac{1}{1+x^2}M\{dx\}$ has $M\{dx\} = M(x)dx$ with

$$M(x) = \frac{1}{8\pi}x^2 \left|\hat{\zeta}\left(\frac{ix}{2}\right)\right|^2 |1-2^{1+\frac{ix}{2}}|^2 \ = \frac{1}{\pi}\xi_{\mathbb{Q}}(0,ix), \tag{1.10}$$

in which $\hat{\zeta}(s)$ is the completed Riemann zeta function. In particular,

$$Z_{\mathbb{Q}}\left(-u, -\frac{u}{2} + it\right) > 0 \qquad \textit{for} \qquad -\infty < t < \infty. \tag{1.11}$$

Many interesting connections between zeta and theta functions and probability theory are known; see Biane, Pitman and Yor [4] for a comprehensive survey. Theorem 1.2 appears structurally different from any of the known results. The positivity property (1.11) can be called an "anti-Riemann hypothesis", because it shows there are no zeros on the "critical line" $\mathrm{Re}(s) = -\frac{u}{2}$ for fixed real $u > 0$.

There are a number of different canonical forms used to specify infinitely divisible distributions. Feller [10, pp. 563] uses the canonical measure $M\{dx\}$, which we term the *Feller canonical measure*, while the *Khintchine canonical measure* $K\{dx\} = \frac{1}{1+x^2}M\{dx\}$ is often used, see [10, pp.564-5]. An infinitely divisible distribution is a member of a positive convolution semigroup of measures, and the Feller canonical measure is related to the infinitesimal generator of the semigroup. The measure $M(x)dx$ above involves the values of the Riemann zeta function on the boundary of its critical strip, noting that the functional equation gives

$$\hat{\zeta}(\frac{ix}{2}) = \hat{\zeta}(1 - \frac{ix}{2}).$$

Theorem 1.2 follows from two results proved in §7, Theorem 7.1 and Theorem 7.4. We also note that the value $\theta(1) = \frac{\pi^{\frac{1}{4}}}{\Gamma(\frac{3}{4})} \approx 1.08643$ appearing in Theorem 1.2 equals e^g where g is the "genus of $\mathbb{Q}$" as defined by van der Geer and Schoof [11], see the appendix.

The positive holomorphic convolution semigroup structure associated to this two-variable zeta function merits further study. It seems an interesting question to determine the generality of this positivity property. All algebraic number fields K have an associated holomorphic convolution semigroup of complex-valued measures, which are real-valued measures on the "critical line". However the positivity fails to hold in general, and perhaps is true only for a few specific number fields, see §9.

We comment on related work. There is precedent for studying two-variable functions given by integrals of the form (1.5) with $\theta(t)$ replaced with some other modular form. Conrey and Ghosh [8, Sect. 5] considered a Fourier integral associated to powers of the modular form $\Delta(\tau)$ of weight 12, which is a cusp form. The integral they consider can be transformed to a constant multiple of the integral

$$\tilde{Z}(w, s) = \int_0^\infty \Delta(it)^{w/24} t^{s/2} \frac{dt}{t}, \tag{1.12}$$

where they take $w = k > 0$, and $s = it$. They note that associated Dirichlet series has an Euler product for $w = 1, 2, 3, 4, 6, 8, 12,$ and 24. Bruggeman [7] studied properties of families of automorphic forms of variable weight; he considers powers of the Dedekind eta function family $\eta(t)^w$ in [7, 1.5.5]. This family appears in (1.12) since $\Delta(\tau) = \eta(\tau)^{24}$, see [7, p. 11].

We now summarize the contents of the paper. In §2 we give the analytic continuation and functional equation for $Z_{\mathbb{Q}}(w, s)$, essentially following Riemann's second proof of the functional equation for $\zeta(s)$. We derive integral formulas for $Z_{\mathbb{Q}}(w, s)$, which converge on $\mathbb{C} \times \mathbb{C}$ off certain hyperplanes.

In §3, as a preliminary to later results, we study the Fourier coefficients $c_m(w)$ of the modular form

$$\theta(it)^w = 1 + \sum_{m=1}^{\infty} c_m(w) e^{-\pi imt}.$$

We show that $(-1)^m m! c_m(-w)$ is a polynomial of degree m with nonnegative integer coefficients. For $w = u$ on the positive real axis we obtain the estimate $|c_m(u)| \le 6um^{\frac{u}{2}+1}$, whose merit is that it is uniform in u. For general $|w| = R$ there is an upper bound

$$|c_m(w)| \le C_0 R^{\frac{R}{2}} e^{\pi\sqrt{Rm}},$$

which follows from classical estimates. We show that the Dirichlet series $\tilde{D}_w(s) = \frac{1}{2w} D_w(s)$ for $w \in \mathbb{C}$ has an Euler product if and only if $w = 0, 1, 2, 4$ and 8.

In §4 we study growth properties of the entire function $\xi_{\mathbb{Q}}(w,s) := \frac{s(s-w)}{2w} Z_{\mathbb{Q}}(w,s)$. We first show that $\xi_{\mathbb{Q}}(w,s)$ is an entire function of order one and infinite type in two complex variables, in the sense that it satisfies the growth bound: There is a constant C_1 such that for any $(w,s) \in \mathbb{C}^2$, if $R = |s| + |w| + 1$, then

$$|\xi_{\mathbb{Q}}(w,s)| \le e^{C_1 R \log R}.$$

Thus any linear slice function $f(s) = \xi_{\mathbb{Q}}(as+b, cs+d)$, has at most $O(R \log R)$ zeros in the disk of radius R as $R \to \infty$, provided it is not identically zero. We then show that for fixed $w \in \mathbb{C}$ and fixed $\sigma \in \mathbb{R}$, the function

$$f_{w,\sigma}(t) := \xi_{\mathbb{Q}}(w, \sigma + it) \quad -\infty < t < \infty,$$

has rapid decrease, is in the Schwartz class $\mathcal{S}(\mathbb{R})$, and is uniformly bounded in vertical strips $\sigma_1 \le \sigma \le \sigma_2$, for finite σ_1, σ_2.

In §5 we treat the case $w = 0$ and prove Theorem 1.1.

In §6 we treat the case when $w = u > 0$ is a fixed positive real number, and study the zeros of $\xi_{\mathbb{Q}}(u,s)$. We show that these zeros are confined to the vertical strip $|\Re(s) - \frac{u}{2}| < \frac{u}{2} + 8$. Then we show that the number $N_u(T)$ of zeros ρ having $|\Im(\rho)| \le T$ has similar asymptotics to that of the Riemann zeta function, namely

$$\frac{1}{2} N_u(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S_u(T),$$

with

$$|S_u(T)| \le C_0(u+1)\log(T+u+2).$$

and the constant C_0 is absolute. The zeros of $\xi_{\mathbb{Q}}(u,s)$ appear to lie on the "critical line" $\Re(s) = \frac{u}{2}$ only for special values $u = 1$ and $u = 2$; we observe that only an infinitesimal fraction of zeros are on this line for $u = 4$ and $u = 8$.

In §7 we consider $\xi_{\mathbb{Q}}(w,s)$ where $w = -u$ is a fixed negative real number ($-u < 0$). We prove Theorem 1.2, that the function $\xi_{\mathbb{Q}}(-u,s)$, which is necessarily real on the critical line $\Re(s) = -\frac{u}{2}$, is always positive there. The proof of this result makes essential use of the Jacobi product formula, which is applicable because the constant term in the theta function is present in the integral representation (1.5). The associated structure behind Theorem 1.2 is a holomorphic convolution semigroup $\rho_{u,v}(x)dx$ of complex-valued measures on the real line, defined for (u,v) real in the cone $u > 0$ and $|v| < u$, and these measures are positive real on the line $v = 0$. We derive formulae for the cumulants and moments of these measures. We also list a number of open questions concerning the location of zeros for negative real u. For example, for real $w = u$, are the asympotics of the number of zeros ρ with $|\Im(\rho)| < T$ as $T \to \infty$ the same for negative real u as they are for positive real u?

In §8 we consider general complex w, and the zero locus $\mathcal{Z}_{\mathbb{Q}}$ of $\xi_{\mathbb{Q}}(w,s)$. The set $\mathcal{Z}_{\mathbb{Q}}$ viewed geometrically † is a one-dimensional complex manifold, having more than one irreducible analytic component (possibly infinitely many components), each one of which is a Riemann surface embedded in $\mathbb{C}^2$. We show that the zeta zeros ρ_7 and ρ_8 are on the same irreducible component, and raise the question whether the zeta zeros (for $w = 1$, s varying) are all on a single irreducible component of the zero locus.

In §9 we briefly consider Arakelov zeta functions attached to general algebraic number fields K. All results of this paper extend to the Arakelov zeta function attached to the Gaussian field $K = \mathbb{Q}(i)$, and many of the results extend to general K, with similar proofs. However the positivity property of Theorem 1.2 for $K = \mathbb{Q}$, our proof used a product formula for the modular form and does not extend to general number fields K. Numerical experiments show that the positivity property does not hold for several imaginary quadratic fields, with discriminants $-8, -11$ and -19. Our computations allows the possibility that it might hold for some fields whose modular forms do not have a product formula, including the imaginary quadratic fields with discriminants -3 and -7.

In the appendix we review the Arakelov divisor framework of van der Geer and Schoof [11], and derive formulas for the two-variable zeta function for $K = \mathbb{Q}$ and $\mathbb{Q}(i)$.

Acknowlegments. The authors thank E. Bombieri, J. B. Conrey, C. Deninger, J. Pitman, J. A. Reeds, M. Yor and the reviewer for helpful comments and references.

Notation. The variables w, s, z denote complex variables with $w = u + iv$, $s = \sigma + it$, $z = x + iy$, and u, v, σ, t, x, y always denote real variables. We use two versions of the Fourier transform, differing in their scaling, because the usual conventions for the Fourier transform differ in probability theory and number theory. The *Fourier transform* $\mathcal{F}$ is given by

$$\mathcal{F}f(x) := \int_{-\infty}^{\infty} f(t)e^{-2\pi ixt}dt,$$

with inverse

$$\mathcal{F}^{-1}f(t) := \int_{-\infty}^{\infty} f(x)e^{2\pi ixt}dx.$$

In probability theory the *characteristic function* $\varphi = \varphi_M$ of a Borel measure $M\{dx\}$ of unit mass on the line is

$$\varphi_M(t) := \int_{-\infty}^{\infty} e^{ixt}M\{dx\},$$

In the case where $M\{dx\} = f(x)dx$ we write $\varphi_M(t) = \check{f}(t)$ as an inverse Fourier transform, and the corresponding Fourier transform is

$$\hat{f}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-ixt}dt. \tag{1.13}$$

†Considered algebraically there is the additional problem of determining the multiplicity of each irreducible component.

2. Analytic Continuation and Functional Equation

We now obtain the meromorphic continuation of $Z_{\mathbb{Q}}(w,s)$, which determines its polar divisor and part of its zero divisor. Using the theta function transformation formula

$$\theta(t^2) = \frac{1}{t}\theta\left(\frac{1}{t^2}\right) \quad \text{for} \quad t > 0 \tag{2.1}$$

we can rewrite

$$Z_{\mathbb{Q}}(w,s) = \int_0^\infty \theta\left(t^2\right)^s \theta\left(\frac{1}{t^2}\right)^{w-s} \frac{dt}{t} \tag{2.2}$$

in the form

$$Z_{\mathbb{Q}}(w,s) = \int_0^\infty \theta\left(\frac{1}{t^2}\right)^w t^{-s}\frac{dt}{t}. \tag{2.3}$$

Then, after a change of variable $t \to 1/t$, followed by $t^2 \to u$, one obtains

$$Z_{\mathbb{Q}}(w,s) = \int_0^\infty \theta(t^2)^w t^s \frac{dt}{t} = \frac{1}{2}\int_0^\infty \theta(u)^w u^{s/2}\frac{du}{u}. \tag{2.4}$$

Note that $\theta(t^2) - 1 \to 0$ rapidly as $t \to \infty$, hence $\theta(t^2) - \frac{1}{t} \to 0$ rapidly as $t \to 0^+$. This implies that (2.2) converges absolutely on the open domain $\mathcal{C} = \{(w,s) : \Re(w) < \Re(s) < 0\}$ in $\mathbb{C}^2$. The convergence is uniform on compact subsets of this domain, which defines $Z_{\mathbb{Q}}(w,s)$ as an analytic function there.

Theorem 2.1. *The function $\xi_{\mathbb{Q}}(w,s) = \frac{s(s-w)}{2w} Z_{\mathbb{Q}}(w,s)$ analytically continues to an entire function on $\mathbb{C}^2$, and satisfies the functional equation*

$$\xi_{\mathbb{Q}}(w,s) = \xi_{\mathbb{Q}}(w, w-s)\ . \tag{2.5}$$

Remark. For $w = 1$ we have $Z_{\mathbb{Q}}(1,s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ and $\xi_{\mathbb{Q}}(1,s) = \xi(s)$ where $\xi(s) := \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$ is Riemann's ξ-function, and we recover the functional equation for $\zeta(s)$. We give a Fourier-Laplace transform integral representation for $\xi_{\mathbb{Q}}(w,s)$ in Theorem 4.4.

Proof. We split the integral (2.2) into two pieces $\int_0^1$ and $\int_1^\infty$ and consider them separately. Using the transformation law yields

$$\begin{aligned}\int_0^1 \theta(t^2)^s \theta\left(\frac{1}{t^2}\right)^{w-s}\frac{dt}{t} &= \int_0^1 \theta\left(\frac{1}{t^2}\right)^w t^{-s}\frac{dt}{t} \\ &= \int_0^1 \left(\theta\left(\frac{1}{t^2}\right)^w - 1\right) t^{-s}\frac{dt}{t} + \int_0^1 t^{-s}\frac{dt}{t}\,. \end{aligned} \tag{2.6}$$

Both sides are defined and converge $\mathrm{Re}(s) < 0$ and $\mathrm{Re}(w) < 0$. On the right side the first integral converges for all $(w,s) \in \mathbb{C} \times \mathbb{C}$, because for $|w| \le R$ and $0 \le t \le 1$,

$$\theta\left(\frac{1}{t^2}\right)^w = 1 + we^{-\frac{\pi}{t^2}} + O\left(e^{-\frac{(2-\epsilon)\pi}{t^2}}\right) \tag{2.7}$$

as $t \to 0^+$, where the constant in the O-symbol depends only on R. This uniformity of convergence shows that this integral is an entire function on $\mathbb{C}^2$. The second integral in (2.6) converges, for $\mathrm{Re}(s) < 0$, to the function $-\frac{1}{s}$.

Similarly,

$$\begin{aligned}
\int_1^\infty \theta(t^2)^s \theta\left(\frac{1}{t^2}\right)^{w-s} \frac{dt}{t} &= \int_0^1 \theta\left(\frac{1}{t^2}\right)^s \theta(t^2)^{w-s} \frac{dt}{t} \\
&= \int_0^1 \theta\left(\frac{1}{t^2}\right)^w t^{s-w} \frac{dt}{t} \\
&= \int_0^1 \left(\theta\left(\frac{1}{t^2}\right)^w - 1\right) t^{s-w} \frac{dt}{t} + \int_0^1 t^{s-w} \frac{dt}{t}\,, \qquad (2.8)
\end{aligned}$$

with both sides convergent for $\mathrm{Re}(s) < 0$, $\mathrm{Re}(w) < 0$ and $\mathrm{Re}(s-w) > 0$. This region overlaps the region of convergence of (2.4) in an open domain in $\mathbb{C}^2$. The first integral on the right side of (2.7) defines an entire function on $\mathbb{C}^2$, while the second integral in (2.8) is $\frac{1}{s-w}$ for $\mathrm{Re}(w-s) > 0$. We obtain

$$Z_{\mathbb{Q}}(w,s) = -\frac{1}{s} + \frac{1}{s-w} + \int_0^1 \left(\theta\left(\frac{1}{t^2}\right)^w - 1\right)(t^{-s} + t^{-(w-s)})\frac{dt}{t} \qquad (2.9)$$

which is valid for $(w,s) \in \mathbb{C}^2$, for $s \neq 0, w$. Since the right side of this equation is invariant under $s \to w-s$, we obtain the functional equation

$$Z_{\mathbb{Q}}(w,s) = Z_{\mathbb{Q}}(w, w-s)\ . \qquad (2.10)$$

Now (2.9) implies that $s(w-s)Z_{\mathbb{Q}}(w,s)$ is an entire function on $\mathbb{C}^2$, and on setting $w = 0$ in (2.9) we see that $Z(0,s)$ is identically zero. Thus $\xi_{\mathbb{Q}}(w,s) = \frac{s(s-w)}{2w} Z_{\mathbb{Q}}(w,s)$ is also an entire function on $\mathbb{C}^2$, and satisfies the same functional equation. ∎

Viewed as a modular form, $\theta(t)$ in (2.2) is not a cusp form, due to its nonzero constant term. One consequence is that the Mellin transform $\int_0^\infty \theta(t) t^s \frac{dt}{t}$ fails to converge anywhere. Riemann's second proof of the functional equation (for $w = 1$) circumvents this problem by removing the constant term, using $2\psi(t) = \theta(t) - 1$ in the integrand, and in this case the Mellin transform integral converges for $\Re(s) > 1$. In Theorem 2.1, the constant term "evaporates" because, formally,

$$\int_0^\infty t^s \frac{dt}{t} \equiv 0\ . \qquad (2.11)$$

More precisely

$$\begin{aligned}
\int_0^1 t^s \frac{dt}{t} &= \frac{1}{s} \quad \text{for} \quad \mathrm{Re}(s) > 0\,, \\
\int_1^\infty t^s \frac{dt}{t} &= -\frac{1}{s} \quad \text{for} \quad \mathrm{Re}(s) < 0\,.
\end{aligned}$$

One convention for "regularization" of the integral is to analytically continue these two pieces separately and then add them, which results in (2.11). Theorem 2.1 justifies this convention by introducing the extra variable w, finding a common domain $\mathcal{C}$ in the (w,s)-plane where the integral converges, and then analytically continuing in both variables to the line $w = 1$.

We next give modified integral formulas for $Z_{\mathbb{Q}}(w,s)$ valid on most of $\mathbb{C}^2$. We define *Heaviside's function* $H(s)$ for complex s with $\Re(s) \neq 0$ to be

$$H(s) = H(\Re(s)) = \begin{cases} 1 & \Re(s) > 0 \\ \frac{1}{2} & \Re(s) = 0 \\ 0 & \Re(s) < 0 \end{cases}.$$

Theorem 2.2. *If $\Re(s) \notin \{\Re(w), 0\}$, then*

$$\begin{aligned} Z_{\mathbb{Q}}(w,s) &= \int_0^\infty (\theta(t^2)^w - H(s) - H(w-s)t^{-w})t^s \frac{dt}{t}, \\ &= \int_{-\infty}^\infty (\theta(e^{2x})^w - H(s) - H(w-s)e^{-wx})e^{sx} dx, \end{aligned} \tag{2.12}$$

where both integrals converge absolutely.

Remark. This result expresses $Z_{\mathbb{Q}}(w,s)$ by a convergent integral formula with integrand

(I)	$\theta(t^2)^w$	if	$\Re(w) < \Re(s) < 0$	(2.13)
(II)	$\theta(t^2)^w - 1$	if	$\Re(w-s) < 0 < \Re(s)$	(2.14)
(III)	$\theta(t^2)^w - 1 - t^{-w}$	if	$0 < \Re(s) < \Re(w)$	(2.15)
(IV)	$\theta(t^2)^w - t^{-w}$	if	$\Re(s) < 0 < \Re(w-s)$	(2.16)

These regions are pictured in Figure 2.1; the dotted line is the "critical line" $\sigma = \frac{\Re(w)}{2}$.

Proof. The second integral (Laplace transform) follows by the change of variable $t = e^x$, so it suffices to consider the first integral. We recall

$$Z_{\mathbb{Q}}(w,s) = -\frac{1}{s} + \frac{1}{s-w} + \int_0^1 (\theta(1/t^2)^w - 1)t^{-s}\frac{dt}{t} + \int_0^1 (\theta(1/t^2)^w - 1)t^{-w+s}\frac{dt}{t} \tag{2.17}$$

$$= -\frac{1}{s} + \frac{1}{s-w} + \int_1^\infty (\theta(t^2)^w - 1)t^s\frac{dt}{t} + \int_0^1 (t^u\theta(t^2)^w - 1)t^{-w+s}\frac{dt}{t}. \tag{2.18}$$

Now, we observe that for $\Re(s) \neq 0$,

$$\int_1^\infty H(-s)t^s\frac{dt}{t} = -\frac{H(-s)}{s} \tag{2.19}$$

$$\int_0^1 H(s)t^s\frac{dt}{t} = \frac{H(s)}{s}; \tag{2.20}$$

whichever integral would diverge is killed by the factor of $H(\pm s)$. By replacing

$$-\frac{1}{s} = -\frac{H(s) + H(-s)}{s} \tag{2.21}$$

$$= \int_1^\infty H(-s)t^s\frac{dt}{t} - \int_0^1 H(s)t^s\frac{dt}{t}, \tag{2.22}$$

and similarly for $\frac{1}{s-w}$, the formula for $Z_{\mathbb{Q}}(w,s)$ simplifies to give the desired result. ∎

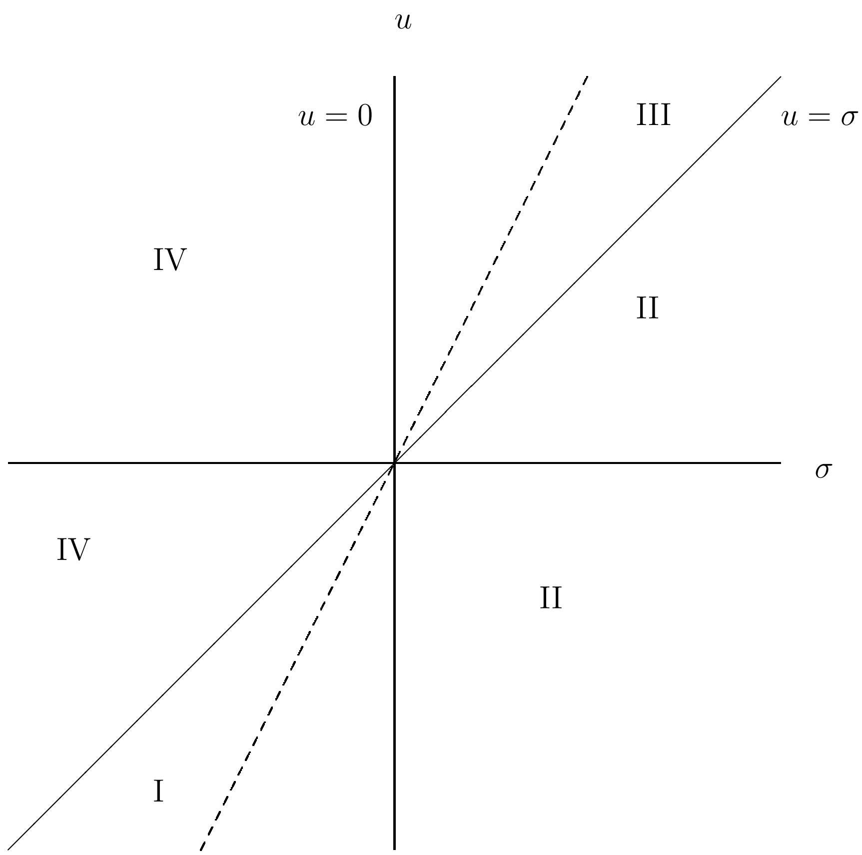

Figure 2.1: Convergence regions: $u = \Re(w), \sigma = \Re(s)$.

Remark. Since $Z_{\mathbb{Q}}(w,s)$ has singularities at $s=0$ and $s=\Re w$ when $w=\Re(w)$ is real, we cannot obtain quite as nice an expression for $\Re(w)=u$ along vertical lines $\Re(s)\in\{0,\Re(w)\}$. Indeed, the Heaviside functions are precisely the contributions of the poles as we move the integral through those points; the poles are also reflected in the fact that for $\Re(s)\in\{0,\Re(w)\}$, the integral diverges. However, if we renormalize the integrals:

$$\int_0^\infty f(t)\frac{dt}{t} \to \lim_{B\to\infty}\frac{1}{\log B}\int_1^B\int_{1/A}^A f(t)\frac{dt}{t}\frac{dA}{A}, \tag{2.23}$$

$$\int_{-\infty}^\infty f(v)dv \to \lim_{B\to\infty}\frac{1}{B}\int_0^B\int_{-A}^A f(v)dv\,dA, \tag{2.24}$$

then the formula is in fact valid for all $s\neq\{0,u\}$; to prove this, use the identity

$$-1/s = \lim_{B\to\infty}\frac{1}{\log B}\int_1^B\int_{1/A}^A \frac{1}{2}\mathrm{sgn}(t-1)t^s\frac{dt}{t}\frac{dA}{A},$$

valid for $\Re(s)=0$, and proceed as before.

3. Fourier Coefficient Estimates

The function $\theta(t)=\sum_{n\in\mathbb{Z}}e^{-\pi n^2 t}$ is a modular form of weight $\frac{1}{2}$ in the variable $\tau=it$ for τ in the upper half-plane, with a multiplier system with respect to the *theta group*‡ Γ_θ, a non-normal subgroup of index 3 in the modular group $PSL(2,\mathbb{Z})$ which contains $\Gamma(2)$, the principal congruence subgroup of level 2. Thus $\theta(t)^w$ is a modular form of (complex) weight $\frac{w}{2}$ with (non-unitary) multiplier system on the same group. We consider its Fourier expansion at the cusp $i\infty$ (of width 2) , given by

$$\theta(it)^w = 1+\sum_{m=1}^\infty c_m(w)e^{-\pi imt}\ . \tag{3.1}$$

(The theta group has two cusps, with the second cusp at -1, see Bruggeman[7, Chap. 14]; we do not consider the other cusp here.) In this section our object is to obtain estimates for the size of the Fourier coefficients $c_m(w)$ as $m\to\infty$ for fixed u. At the end of the section we give explicit formulas for a few integer values of w where the Fourier coefficients have arithmetic significance, namely $w=0,1,2,4,6,$ and 8.

3.1. Fourier Coefficient Formulas

We establish basic properties of the Fourier coefficients $c_m(w)$ as a function of $w=u+iv$.

Theorem 3.1. *The Fourier coefficient $c_m(w)$ is a polynomial in $\mathbb{Q}[w]$ of degree m. For each $m\geq 1$, the polynomial*

$$c_m^*(w) := (-1)^m m!\ c_m(-w),\quad m\geq 1\,, \tag{3.2}$$

has nonnegative integer coefficients, lead term $2^m w^m$, and vanishing constant term.

$^\ddagger\Gamma_\theta$ is the set of $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\equiv\begin{pmatrix}1&0\\0&1\end{pmatrix}$ or $\begin{pmatrix}0&1\\1&0\end{pmatrix}$ (mod 2) in $PSL(2,\mathbb{Z})$.

To prove this result we will need the triple product formula of the Jacobi theta function $\vartheta_3(z,q)$, see Andrews [1, Theorem 2.8] or Andrews, Askey and Roy [2, Section 10.4].

Proposition 3.2. (*Jacobi Triple Product Formula*) *The Jacobi theta function*

$$\vartheta_3(z,q) := \sum_{n\in\mathbb{Z}} e^{2\pi i n z} q^{n^2}$$

is given by

$$\vartheta_3(z,q) = \prod_{n=1}^{\infty}(1-q^{2n})(1+e^{2\pi i z}q^{2n-1})(1+e^{-2\pi i z}q^{2n-1}). \tag{3.3}$$

This formula is valid for $|q|<1$ and all $z \in \mathbb{C}$.

Proof of Theorem 3.1. The Fourier coefficients $c_m(w)$ are computable using the expansion

$$\begin{aligned}\theta(t)^w &= \left(1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 t}\right)^w \\ &= 1 + \sum_{j=1}^{\infty}\binom{w}{j}2^j\left(\sum_{n=1}^{\infty} e^{-\pi n^2 t}\right)^j .\end{aligned} \tag{3.4}$$

Terms involving $e^{-\pi m t}$ can appear only for $1 \le j \le m$, hence we find that $c_m(w)$ is a polynomial of degree m in w with rational coefficients and leading term $\frac{2^m}{m!}w^m$. Clearly

$$c_1(w) = 2w \tag{3.5}$$

and $c_2(w) = 2w(w-1)$. Multiplication by $m!$ clears denominators $m!\binom{w}{j} \in \mathbb{Z}[w]$ for $1 \le j \le m$ hence $c_m^*(w) = (-1)^m m! c_m(w) \in \mathbb{Z}[w]$.

It remains to show nonnegativity. We have

$$\vartheta_3(0,-q)^{-w} = 1 + \sum_{m=1}^{\infty}\frac{c_m^*(w)}{m!}q^m . \tag{3.6}$$

The Jacobi triple product formula gives

$$\vartheta_3(0,q) = \prod_{k=1}^{\infty}(1+q^{2k-1})^2(1-q^{2k}) \tag{3.7}$$

hence

$$\vartheta_4(0,q) = \vartheta_3(0,-q) = (1-q)\prod_{k=1}^{\infty}(1-q^{2k-1})(1-q^{2k})(1-q^{2k+1})\,.$$

For $w = u > 0$ real, we have

$$\vartheta_3(0,-q)^{-u} = (1-q)^{-u}\prod_{k=1}^{\infty}(1-q^{2k-1})^{-u}(1-q^{2k})^{-u}(1-q^{2k+1})^{-u}\,. \tag{3.8}$$

Now

$$(1-z)^{-w} = 1 + \sum_{k=1}^{\infty} \binom{w+l-1}{l} z^l \tag{3.9}$$

$$= 1 + \sum_{l=1}^{\infty} \sum_{m=0}^{l} c_{lm} w^m z^l \tag{3.10}$$

is a bivariate power series with all $c_{lm} \geq 0$. This nonnegativity property is preserved under multiplication of power series, hence $\vartheta_3(0,-q)^{-u}$ inherits this property by (3.8). Thus all the coefficients of $c_m^*(w)$ are nonnegative. One has

$$c_m^*(w) = \sum_{k=0}^{m} c_{mk}^* w^k \ , \tag{3.11}$$

in which $c_{m0}^* = 0$ and $c_{mk}^* > 0$ for $1 \leq k \leq m$, with $c_{mm}^* = 2^m$. ■

3.2. Fourier coefficients for positive real w

Let $w = u \in \mathbb{R}_{>0}$. In this case $\theta(t)^u$ is a modular form of real weight $u/2$, on the theta group Γ_θ, with a unitary multiplier system. Classical estimates of Petersson [20] and Lehner [16] for the Fourier coefficients of arbitrary modular forms of positive real weight (with multiplier systems) show they grow polynomially in m, with

$$c_m(u) = O(m^{u/2-1}) \quad \text{if} \quad u > 4\,,$$

with $c_m(u) = O(m^{u/2-1}\log m)$ for $u = 4$ and $c_m(u) = O(m^{u/4})$ for $0 < u < 4$, where the O-symbol constants depend on u in an unspecified manner.

Here we establish some weaker estimates, whose merit is that the dependence on u is completely explicit, for use in §6.

Theorem 3.3. *Suppose $w = u \geq 0$ is real.*

(i) For $m \geq 2$,

$$|c_m(u)| \leq 24 m^{\frac{u}{2}} \ , \tag{3.12}$$

(ii) For $m \geq 1$,

$$|c_m(u)| \leq 6u m^{\frac{u}{2}+1} \ . \tag{3.13}$$

Proof. (i) Write $\vartheta_3(q) = \vartheta_3(0,q) = \sum_{n\in\mathbb{Z}} q^{n^2}$, so that $\theta(t) = \vartheta_3(e^{-\pi t})$. Using Cauchy's theorem we obtain the following formula:

$$\begin{aligned} c_m(u) &= \frac{1}{2\pi i} \oint \vartheta_3(q)^u q^{-m-1} dq \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \vartheta_3(Re^{i\theta})^u R^{-m} e^{-im\theta} d\theta \\ &\leq R^{-m} \big(\max_{-\pi\leq\theta\leq\pi} |\vartheta_3(Re^{i\theta})^u| \big), \end{aligned} \tag{3.14}$$

for any choice $0 < R < 1$. We take $R = e^{-1.01\frac{\pi}{m}}$, and thus

$$R^{-m} = e^{1.01\pi} < 24.$$

For the other term, we first observe that, since $u \geq 0$,

$$\max_{-\pi\leq\theta\leq\pi} |\vartheta_3(Re^{i\theta})^u| = (\max_{-\pi\leq\theta\leq\pi} |\vartheta_3(Re^{i\theta})|)^u.$$

Since the coefficients of $\vartheta_3(q)$ are positive, the maximum occurs for $\theta = 0$; thus we must estimate $\vartheta_3(R)$. Using the functional equation of $\vartheta_3(q)$, we find, for $m \geq 2$:

$$\begin{aligned}
\vartheta_3(R) &= \vartheta_3(e^{-1.01\frac{\pi}{m}}) \\
&= \sqrt{\frac{m}{1.01}}\vartheta_3(e^{-\frac{\pi m}{1.01}}) \\
&\leq \sqrt{m},
\end{aligned} \tag{3.15}$$

using

$$\vartheta_3(e^{-\frac{2\pi}{1.01}}) < \sqrt{1.01}.$$

The bound (3.12) follows immediately.

(ii) For $m = 1$ we have $c_1(u) = 2u \leq 6u$, so we may restrict our attention to $m \geq 2$. Differentiating $\vartheta_3(q)^u$ by $\frac{\partial}{\partial q}$, (denoted by $'$) and dividing by u, we find:

$$q\vartheta_3'(q)\vartheta_3(q)^{u-1} = \sum_{m\geq 0} \frac{mc_m(u)}{u} q^m.$$

Using Cauchy's theorem we obtain the following formula:

$$\begin{aligned}
m\frac{c_m(u)}{u} &= \frac{1}{2\pi i}\int_{|q|=R} q\vartheta_3'(q)\vartheta_3(q)^{u-1}\frac{dq}{q^{m+1}} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{Re^{i\theta}\vartheta_3'(Re^{i\theta})}{\vartheta_3(Re^{i\theta})}\vartheta_3(Re^{i\theta})^u R^{-m}e^{-im\theta}d\theta \\
&\leq R^{-m}(\max_{-\pi\leq\theta\leq\pi} |\vartheta_3(Re^{i\theta})^u|)(\max_{-\pi\leq\theta\leq\pi} \left|\frac{Re^{i\theta}\vartheta_3'(Re^{i\theta})}{\vartheta_3(Re^{i\theta})}\right|),
\end{aligned} \tag{3.16}$$

Taking $R = e^{-1.01\frac{\pi}{m}}$ as before, we need only estimate the third factor.

The Jacobi triple product formula implies that the coefficients of $q\frac{\vartheta_3'(q)}{\vartheta_3(q)}$ are alternating in sign; in other words, using $\vartheta_4(q) = \vartheta_3(-q)$ the power series expansion in q of

$$-q\frac{\vartheta_4'(q)}{\vartheta_4(q)}$$

has positive coefficients. In particular, its maximum is given by

$$\begin{aligned}
-R\frac{\vartheta_4'(R)}{\vartheta_4(R)} &= \frac{m^2}{(1.01)^2}e^{-\frac{\pi m}{1.01}}\frac{\vartheta_2'(e^{-\frac{\pi m}{1.01}})}{\vartheta_2(e^{-\frac{\pi m}{1.01}})} - \frac{m}{2.02\pi} \\
&\leq \frac{m^2}{(1.01)^2}e^{-\frac{\pi m}{1.01}}\frac{\vartheta_2'(e^{-\frac{\pi m}{1.01}})}{\vartheta_2(e^{-\frac{\pi m}{1.01}})}.
\end{aligned} \tag{3.17}$$

Writing

$$\frac{q\vartheta_2'(q)}{\vartheta_2(q)} = \sum_{k\geq 0} e_k q^k,$$

we estimate

$$\begin{aligned} e^{-\frac{\pi m}{1.01}} \frac{\vartheta_2'(e^{-\frac{\pi m}{1.01}})}{\vartheta_2(e^{-\frac{\pi m}{1.01}})} &\leq \sum_{k\geq 0} |e_k| e^{-\frac{\pi k m}{1.01}} \\ &\leq \sum_{k\geq 0} |e_k| e^{-\frac{2\pi k}{1.01}} = .25000790 \end{aligned} \tag{3.18}$$

We conclude that

$$-R\frac{\vartheta_4'(R)}{\vartheta_4(R)} \leq .24508175 < 1/4.$$

Substituting this in (3.16), we conclude

$$m\frac{c_m(u)}{u} \leq 6m^{u/2+2}.$$

Now (3.13) follows after multiplying by $\frac{u}{m}$. ∎

Theorem 3.3 implies that for real positive $w = u$ the Dirichlet series

$$D_u(s) = \sum_{m=1}^{\infty} c_m(u) m^{-s} \tag{3.19}$$

converges absolutely on the half-plane $\Re(s) > \frac{u}{2} + 1$. The estimate of Petersson [20] for the Fourier coefficients implies that the Dirichlet series converges absolutely in the half-plane $\Re(s) > \frac{u}{2}$ for $u > 4$.

It seems likely that for general $w \in \mathbb{C}$ the Dirichlet series has no half-plane of absolute convergence, except when $w = u \in \mathbb{R}_{\geq 0}$, because the Fourier coefficients grow too rapidly off the positive real axis. We do not address this question in this paper.

3.3. Maximum Size of Fourier Coefficients

By Theorem 3.1 the maximium size of $c_m(u)$ on the circle $|u| = R$ occurs on the negative real axis $u = -R$. Convergent series are known for Fourier coefficients of modular forms of negative real integer weight, see Petersson [19] and Lehner [15, Theorem 1]; the convergent series of Rademacher for the partition function is an example. The following proposition extracts the main term in that convergent expansion, as given in Lehner [15, Theorem 1]; one can also prove it following the proof for the partition function in Apostol [3, Chap. 5].

Proposition 3.4. *For real $-u < 0$, there holds*

$$c_m(-u) = (-1)^m 2\pi u^{\frac{u}{2}} \frac{1}{(4m)^{u+1}} I_{u+1}(\pi\sqrt{um}) + O(e^{(\pi+\epsilon)\sqrt{max(\frac{um}{9},(u-8)m)}}), \tag{3.20}$$

where $I_\alpha(x)$ is the modified Bessel function of the first kind, given by

$$I_\alpha(x) = \left(\frac{x}{2}\right)^\alpha \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2k}}{k!\Gamma(\alpha+k+1)}. \tag{3.21}$$

For fixed $\alpha > 0$, this function satisfies

$$I_\alpha(x) = \frac{e^x}{\sqrt{2\pi x}}\left(1 + O_\alpha\left(\frac{1}{x}\right)\right) \quad as \quad x \to \infty\,. \tag{3.22}$$

The formula (3.20) implies a general upper bound of the form

$$|c_m(w)| \le C_0 R^{\frac{R}{2}} e^{\pi\sqrt{Rm}}\,. \tag{3.23}$$

for all complex $|w| = R$.

3.4. Euler Products

For a few special values of w the associated Dirichlet series has an Euler product, and the Fourier coefficients have an explicit description. For $w \in \mathbb{C}$ we define the function $\tilde{D}_w(\frac{s}{2})$ by

$$\xi_{\mathbb{Q}}(w,s) = \frac{1}{2}s(s-w)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\tilde{D}_w(\frac{s}{2}).$$

The function $\tilde{D}_w(s)$ can be assigned a (formal) Dirichlet series which for $w \neq 0$ is

$$\tilde{D}_w(s) := \frac{1}{2w}D_w(s) \sim 1 + \sum_{m=2}^{\infty} \tilde{c}_m(w)m^{-s},$$

and for $w = 0$ is

$$\tilde{D}_0(s) = 1 + \sum_{m=2}^{\infty} \tilde{c}_m(0)m^{-s},$$

in which $\tilde{c}_m(0) = \frac{c_m(w)}{2w}|_{w=0} = \frac{1}{2}\frac{d}{dw}c_m(w)|_{w=0}$. Here the use of $\sim$ indicates that the (formal) Dirichlet series expansion based on Fourier coefficients need not have any region of absolute convergence. However for real nonnegative w it does converge on a half-plane, as follows from Theorem 3.3.

It is easy to determine which values w give (formal) Dirichlet series that have an Euler product.

Lemma 3.5. *For $w \in \mathbb{C}$ the formal Dirichlet series assigned to $\tilde{D}_w(s)$ has an Euler product expansion if and only if $w = 0, 1, 2, 4$ and 8.*

Proof. To see that $w = 0, 1, 2, 4$, and 8 are the only complex values for which $\tilde{D}_w(s)$ can have an Euler product, we consider the necessary conditon

$$\tilde{c}_2(w)\tilde{c}_3(w) = \tilde{c}_6(w).$$

This gives a polynomial of degree five in w whose roots are $w = 0, 1, 2, 4$, and 8.

For $w = 1, 2, 4$ and 8 the Dirichlet series $\tilde{D}$ has an Euler product. For $w = 1$ we have already seen that

$$\tilde{D}_1(s) = \zeta(2s) = \prod_p (1 - p^{-2s})^{-1}.$$

For $w = 2$,

$$\tilde{D}_2(s) = \sum_{m=1}^{\infty} (-4/m) m^{-s} = (1 - 2^{-s})^{-1} \prod_{p=1(4)} (1 - p^{-s})^{-1} \prod_{p=3(4)} (1 + p^{-s})^{-1}.$$

For $w = 4$,

$$\begin{aligned} \tilde{D}_4(s) &= (1 - 4 \cdot 2^{-2s}) \zeta(s) \zeta(s-1) \\ &= \frac{1 + 2 \cdot 2^{-s}}{1 - 2^{-s}} \prod_{p \ odd} (1 - p^{-s})^{-1} (1 - p \cdot p^{-s})^{-1}. \end{aligned}$$

For $w = 8$,

$$\begin{aligned} \tilde{D}_8(s) &= (1 - 2 \cdot 2^{-s} + 16 \cdot 2^{-2s}) \zeta(s) \zeta(s-3) \\ &= \frac{1 - 2 \cdot 2^{-s} + 16 \cdot 2^{-2s}}{(1 - 2^{-s})(1 - 8 \cdot 2^{-s})} \prod_{p \ odd} (1 - p^{-s})^{-1} (1 - p^3 \cdot p^{-s})^{-1}. \end{aligned}$$

These Dirichlet series are scaled multiples of the zeta functions for the rational field, the Gaussian integers, the integral quaternions and the integral octaves, respectively.

For $w = 0$ the Dirichlet series also has an Euler product, which is

$$\tilde{D}_0(s) = \frac{1 - 2^{1-s}}{1 - 2^{-s}} \prod_{p \ odd} (1 - p^{-s})^{-1} (1 - p^{-1-s})^{-1}. \tag{3.24}$$

This follows from Theorem 5.1 below; note that it converges absolutely for $\Re(s) > 1$. ∎

One immediately sees from the expression for $\xi_{\mathbb{Q}}(4, s)$ as a product of shifted Riemann zeta functions that most of its zeros cannot be on "critical line" $\Re(s) = 2$. The only zeros on the line come from the Euler factor at the prime 2 and have are of number $O(T)$ up to height T, while there are $\Omega(T \log T)$ zeros off the line coming from the Riemann zeta zeros. Exactly the same thing happens for $\xi_{\mathbb{Q}}(8, s)$.

There are other positive integer values of w where the Dirichlet series can be determined explicitly, without an Euler product. For $w = 6$ the Dirichlet series is a linear combination of two Dirichlet series with Euler products, namely

$$\tilde{D}_6(s) = \frac{4}{3} \zeta(s-2) L(s, \chi_{-4}) - \frac{1}{3} \zeta(s) L(s-2, \chi_{-4}),$$

in which $L(s, \chi_{-4}) = \sum_{m=1}^{\infty} (-4/m) m^{-s}$, with $(-4/m)$ being the Jacobi symbol.

4. Growth Bounds

It is well known that the Riemann ξ-function $\xi(s) = \frac{1}{2} s(s-1) \pi^{\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s)$ is an entire function of order one and infinite type. It is bounded in vertical strips. In this section we show that both these properties generalize to the function $\xi_{\mathbb{Q}}(w, s)$.

4.1. Growth of Maximum Modulus

We prove the following bound for the the two-variable zeta function.

Theorem 4.1. *There is a constant $C > 0$ such that the entire function $\xi_{\mathbb{Q}}(w, s)$ satisfies the growth bound: For all $(w, s) \in \mathbb{C}^2$, if $R = |w| + |s| + 1$ then*

$$|\xi_{\mathbb{Q}}(w, s)| \leq e^{CR\log R}. \tag{4.1}$$

Proof. Set

$$I(w, s) = \int_0^1 \frac{\theta(\frac{1}{t^2})^w - 1}{w} t^{-s} \frac{dt}{t} = \frac{1}{2} \int_1^\infty \frac{\theta(t)^w - 1}{w} t^{s/2} \frac{dt}{t}$$

Then (2.9) gives

$$2\xi_{\mathbb{Q}}(w, s) = 1 + s(s - w)(I(w, s) + I(w, w - s)).$$

For $t > 0$,

$$\theta(t) = 1 + 2 \sum_{m \geq 1} e^{-\pi m^2 t},$$

and $\theta(t) > 1$ for all $t > 0$ since all of the terms are positive. It follows that for all positive t and all $w \in \mathbb{C}$,

$$\left| \frac{\theta(t)^w - 1}{w} \right| \leq \frac{\theta(t)^{|w|} - 1}{|w|}.$$

Moreover, the right-hand-side is increasing in $|w|$, so we find that for all $t > 0$,

$$\left| \frac{\theta(t)^w - 1}{w} \right| \leq \frac{\theta(t)^R - 1}{R} \leq \theta(t)^R \log \theta(t),$$

using $\frac{e^{Rx}-1}{R} \leq x e^{Rx}$ for $x \geq 0$.

Similarly, the function $e^{\pi t}(\theta(t) - 1)$ is decreasing in t, so we find that for $1 \leq t < \infty$, we have:

$$\theta(t) \leq 1 + C_1 e^{-\pi t} \leq e^{C_1 e^{-\pi t}},$$

where $C_1 = e^{\pi}(\theta(1) - 1)$. In particular, for $t \geq 1$,

$$\theta(t)^R \log \theta(t) \leq e^{C_1 e^{-\pi t} R} C_1 e^{-\pi t} = C_1 e^{C_2 R} e^{-\pi t}.$$

We thus have

$$\begin{aligned}
|I(w, s)| &\leq \frac{1}{2} \int_1^\infty \left| \frac{\theta(t)^w - 1}{w} \right| |t^s| \frac{dt}{t} \\
&\leq \frac{1}{2} C_1 e^{C_2 R} \int_1^\infty e^{-\pi t} t^{\frac{|s|}{2}} \frac{dt}{t} \\
&\leq \frac{1}{2} C_1 e^{C_2 R} \int_0^\infty e^{-\pi t} t^{\frac{R}{2}} \frac{dt}{t} \\
&= \frac{1}{2} C_1 e^{C_2 R} \pi^{-\frac{R}{2}} \Gamma(\frac{R}{2}).
\end{aligned} \tag{4.2}$$

Since $R \geq 1$, and $\Gamma(R) \leq R^R$ on this range, it follows that there exists a constant $C_3 > 0$ such that

$$|I(w,s)| \leq e^{C_3 R \log R}.$$

Since both 1 and $|s(s-w)|$ are $o(R^R)$, the theorem follows. ∎

A notion of entire function of finite order for functions of several complex variables is described in Ronkin [23], [24] and Stoll [25]. In particular, there is a Weierstrass factorization theorem for such functions in terms of their zero locus.

The zero locus $\mathcal{Z}_{\mathbb{Q}}$ of $\xi_{\mathbb{Q}}(w,s)$ is a one-dimensional complex analytic manifold, possibly with singular points, which may have many connected components. One way to study it is to take "linear slices" to obtain functions $h(s)$ of one complex variable, whose zero sets consist of isolated points which can be counted. If $\alpha := (\alpha_1, \alpha_2)$ and $\beta := (\beta_1, \beta_2)$ with $\alpha_j, \beta_j \in \mathbb{C}$, the *linear slice function* $h_{\alpha,\beta}(s)$ of $\xi_{\mathbb{Q}}(w,s)$, is

$$h_{\alpha,\beta}(s) := \xi_{\mathbb{Q}}(\alpha_1 s + \alpha_2, \beta_1 s + \beta_2) \ , \tag{4.3}$$

where we assume $|\alpha_1| + |\beta_1| \neq 0$ to avoid constant functions.

Lemma 4.2. *Any linear slice function $h_{\alpha,\beta}(s)$ is an entire function of order at most 1. If $N^*_{\alpha,\beta}(T) = \#\{$zeros of $h_{\alpha,\beta}(s)$ with $|s| \leq T\}$, then*

$$N^*_{\alpha,\beta}(T) = O(T \log T) \quad \textit{for} \quad T \geq 2 \ , \tag{4.4}$$

where the implied constant in the O-symbol depends on α, β.

Proof. This follows from the growth estimate in Theorem 4.1, using Jensen's formula. ∎

This result applies in particular to linear slices where $w \in \mathbb{C}$ is held fixed, i.e. $\alpha_1 = 0$, $\alpha_2 = w$, $\beta_1 = 1$ and $\beta_2 = 0$. which gives the function $\xi_{\mathbb{Q}}(w,s)$, with w regarded as constant, e. g. in §7.

4.2. Growth Bounds on Vertical Lines

We next consider growth bounds for $\xi_{\mathbb{Q}}(w,s)$ on vertical lines $s = \sigma + it$ with σ held fixed. Recall that a function $f(x)$ is in the *Schwartz space* $\mathcal{S}(\mathbb{R})$ if and only if for each $m, n \geq 0$ there is a finite constant $C_{m,n}$ such that

$$\sup_{x \in \mathbb{R}} |x^n \frac{d^m}{dx^m} f(x)| \leq C_{m,n},$$

with a similar definition for functions defined on a closed half-line $\mathbb{R}_{\geq 0}$ or $\mathbb{R}_{\leq 0}$.

Theorem 4.3. *For each $w \in \mathbb{C}$, $\sigma \in \mathbb{R}$, and real $-\frac{\pi}{4} < y < \frac{\pi}{4}$, the function*

$$e^{yt} \xi_{\mathbb{Q}}(w, \sigma + it) \qquad -\infty < t < \infty,$$

belongs to the Schwartz space $\mathcal{S}(\mathbb{R})$. Furthermore, the implied Schwartz constants can be chosen uniformly on any compact subset Ω of $\{(w,\sigma,y) : w \in \mathbb{C}, \sigma \in \mathbb{R}, -\frac{\pi}{4} < y < \frac{\pi}{4}\}$. In particular, these functions are bounded in vertical strips, i.e. there is a finite constant $A_1 = A_1(\Omega)$ such that

$$\sup_{(w,\sigma,y) \in \Omega, \ t \in \mathbb{R}} |e^{yt} \xi_{\mathbb{Q}}(w, \sigma + it)| \leq A_1.$$

We will deduce this result from an integral representation of $\xi_{\mathbb{Q}}(w, \sigma + it)$, which we prove first. We define the function $h(w, z)$ for $w \in \mathbb{C}$ and $z = x + iy \in \mathbb{C}$ with $-\frac{\pi}{4} < y < \frac{\pi}{4}$, by

$$h(w, z) := \begin{cases} \frac{\theta(e^{2z})^w - 1}{w} & w \neq 0 \\ \log \theta(e^{2z}) & w = 0. \end{cases}$$

Next we define

$$\gamma(w, z) = (\frac{d^2}{dz^2} + w\frac{d}{dz})h(w, z). \tag{4.5}$$

We have the following Fourier-Laplace transform formula for $\xi_{\mathbb{Q}}(w, s)$, valid on $\mathbb{C}^2$.

Theorem 4.4. *Let $w \in \mathbb{C}$, and real y with $-\frac{\pi}{4} < y < \frac{\pi}{4}$. Then for all $s \in \mathbb{C}$,*

$$\xi_{\mathbb{Q}}(w, s) = \frac{1}{2}\int_{-\infty}^{\infty} \gamma(w, x + iy)e^{xs}dx.$$

Regarded as a Fourier transform, with $s = \sigma + it$ with fixed $\sigma \in \mathbb{R}$, the integrand

$$f_{w,\sigma,y}(x) := e^{\sigma x}\gamma(w, x + iy), \qquad -\infty < x < \infty,$$

belongs to the Schwartz space $\mathcal{S}(\mathbb{R})$ and the implied Schwartz constants are uniform on compact subsets of $\{(w, \sigma, y) : w \in \mathbb{C}, \sigma \in \mathbb{R}, -\frac{\pi}{4} < y < \frac{\pi}{4}\}$.

To prove Theorem 4.4 we formulate several estimates as preliminary lemmas.

Lemma 4.5. *Define an analytic function $f(w, \tau)$ for $w, \tau \in \mathbb{C}$, with $\Re(\tau) > 0$, by*

$$f(w, \tau) = \begin{cases} \frac{\theta(\tau)^w - 1}{w} & w \neq 0 \\ \log \theta(\tau) & w = 0. \end{cases}$$

For any integer $k \geq 0$, and for any region of the form

$$\Omega_{R,\epsilon} = \{(w, \tau) : \ |w| \leq R \text{ and } \Re(\tau) \geq \epsilon\},$$

there is a finite constant $C_k = C_k(R, \epsilon)$ such that

$$|e^{\pi\tau}\frac{d^k}{d\tau^k}f(w, \tau)| \leq C_k.$$

Proof. For $w \neq 0$ the results of §3 show that $f(w, \tau)$ has a Fourier expansion:

$$f(w, \tau) = \sum_{m=1}^{\infty} \frac{c_m(w)}{w}e^{-m\pi\tau}.$$

Note that $\frac{c_m(w)}{w}$ is well-defined as a polynomial in w, hence this expansion makes sense for $w = 0$ as well, with Fourier coefficient $\frac{d}{dw}c_m(w)|_{w=0}$. Now, for $m \geq 1$, we have the inequality

$$\left|\frac{c_m(w)}{w}\right| \leq \frac{(-1)^m c_m(-|w|)}{|w|};$$

this follows from the fact that $(-1)^m c_m(-w)$ has positive coefficients, and is 0 at $w = 0$. Moreover, this upper bound is monotonically increasing in $|w|$. In particular, since $\vartheta_3(q)$ has radius of convergence 1, and no zeros in the unit disk, we find that for fixed $|w| > 0$,

$$\frac{\vartheta_3(q)^{-|w|} - 1}{|w|}$$

has radius of convergence 1, and thus

$$\lim_{m\to\infty} \left|\frac{c_m(w)}{w}\right|^{1/m} = 1.$$

In particular, since $q = e^{-\pi\tau}$, the condition $\Re(\tau) > 0$ makes $f(w,\tau)$ analytic in the region $\Omega_{R,\epsilon}$.

Similarly, for the kth derivative, we have:

$$\frac{d^k}{dt^k} f(w,\tau) = \sum_{m=1}^{\infty} (-\pi m)^k \frac{c_m(w)}{w} e^{-m\pi\tau},$$

and, using term-by-term absolute value estimates,

$$|e^{\pi\tau} \frac{d^k}{d\tau^k} f(w,\tau)| \le \sum_{m=1}^{\infty} (\pi m)^k \frac{(-1)^m c_m(-|w|)}{|w|} e^{-(m-1)\pi\Re(\tau)}.$$

Over the region $|w| \le R$, $\Re(\tau) \ge \epsilon$, this is bounded by its value for $w = R$, $\tau = \epsilon$; as the sum converges for those values, we obtain the required uniform bound. ∎

Lemma 4.6. *Let $z = x + iy$, and for fixed $-\frac{\pi}{4} < y < \frac{\pi}{4}$ and fixed $\sigma \in \mathbb{R}$, the function*

$$f_{w,\sigma,y}(x) = e^{\sigma(x+iy)}\gamma(w, x+iy) \qquad -\infty < x < \infty,$$

where $\gamma(w,z)$ is defined as in Theorem 4.4, lies in the Schwartz space $\mathcal{S}(\mathbb{R})$. Moreover, the implied Schwartz bounds are uniform over compact regions in $\{(w,\sigma,y) : w \in \mathbb{C}, \sigma \in \mathbb{R}, -\frac{\pi}{4} < y < \frac{\pi}{4}\}$.

Proof. By definition $h(w,z) = f(w, e^{2z})$, in which $-\frac{\pi}{4} < y = \Im(z) < \frac{\pi}{4}$ so that $\Re(\tau) = \Re e^{2z} > 0$. We claim that for any $\sigma \in \mathbb{R}$, the function $\{e^{\sigma(x+iy)} h(w, x+iy) \ : \ 0 \le x < \infty\}$ is a Schwartz function on the half-line $\mathcal{S}(\mathbb{R}_{\ge 0})$, and that the implied Schwartz bounds are uniform over compact regions in $w \in \mathbb{C}$, $-\frac{\pi}{4} < y < \frac{\pi}{4}$, $\sigma \in \mathbb{R}$. To see this, note that for $\Re(z) \ge 0$, we find $\Re(e^{2z}) \ge \cos(2\Im(z))$, and since $\Im(z)$ is bounded away from $\pm\pi/4$, $\Re(e^{2z})$ is bounded away from 0. Now using Lemma 4.5, we have

$$|\Re(z)^n \frac{d^m}{dz^m} e^{\sigma z} f(w, e^{2z})| = \left| |\log(\tau)|^n (\tau \frac{d}{d\tau})^m \tau^\sigma f(w,\tau^2) \right|_{\tau = e^z} = O(e^{(\sigma+m)z} z^n e^{-\pi e^z}) = o(1),$$

as $x = \Re(z) \to \infty$, uniformly in a compact region in w, $-\frac{\pi}{4} < \Im(z) < \frac{\pi}{4}$, σ, proving the claim. (The function $|f(w, e^{2z})|$ and its derivatives go to zero at a super-exponential rate as $x \to \infty$, all other variables fixed.)

Now applying the operator $\frac{d}{dz}(\frac{d}{dz} + w)$ to $h(w,z)$, the claim implies that $\gamma(w, x+iy)$ is a Schwartz function in x on $\mathbb{R}_{\ge 0}$. The functional equation $\theta(e^{2z}) = e^{-z}\theta(e^{-2z})$ then yields

$$\gamma(w, -z) = e^{wz}\gamma(w,z),$$

so that $\gamma(w, x+iy)$ is a Schwartz function in x on $\mathbb{R}_{\leq 0}$ as well. Thus it is in $\mathcal{S}(\mathbb{R})$, and the uniformity of the estimates in compact regions in w, $-\frac{\pi}{4} < \Im(z) < \frac{\pi}{4}$ is inherited. ∎

To prove Theorem 4.4, we will use repeated integration by parts starting from the integral representation for $Z_{\mathbb{Q}}(w, s)$ in Theorem 2.2. To justify this step we show that the integrand of that representation is a Schwarz function for $\Re(s) \notin \{0, \Re(w)\}$.

Lemma 4.7. *Let $w, z \in \mathbb{C}$, and $\sigma \in \mathbb{R}$ be fixed, with $\sigma \notin \{0, \Re(w)\}$. Then for $z = x+iy$, and $-\frac{\pi}{4} < y < \frac{\pi}{4}$, the functions of $x \in \mathbb{R}$,*

$$g_\sigma(w, x+iy) = \begin{cases} \frac{e^{\sigma(x+iy)}}{w}(\theta(e^{2(x+iy)})^w - H(\sigma) - H(w-\sigma)e^{-w(x+iy)}) & w \neq 0 \\ e^{\sigma(x+iy)}(\log\theta(e^{2(x+iy)}) - H(\sigma) - H(-\sigma)(x+iy)) & w = 0 \end{cases}$$

all belong to the Schwartz space $\mathcal{S}(\mathbb{R})$.

Proof. It suffices to show that $g_\sigma(w, x+iy)$ and $g_\sigma(w, -x-iy)$ are Schwartz functions on the half-line $x \geq 0$. We have

$$g_\sigma(w, x+iy) = e^{\sigma(x+iy)} f(w, e^{2(x+iy)}) + \frac{1}{w}(H(-\sigma)e^{\sigma(x+iy)} - H(w-\sigma)e^{(\sigma-w)(x+iy})$$

using the relation $1 - H(\sigma) = H(-\sigma)$. We also have

$$g_\sigma(w, -x-iy) = e^{(w-\sigma)(x+iy)} f(w, e^{2(x+iy)}) + \frac{1}{w}(-H(\sigma)e^{-\sigma(x+iy)} + H(\sigma-w)e^{(w-\sigma)(x+iy)}),$$

using the functional equation for $\theta(e^{2z})$. In each case, the first term has been shown to be Schwartz, and to be uniform in the parameters, in the proof of Lemma 4.6. It remains only to consider the "correction" terms, on the half-line $x \geq 0$.

Suppose $w \neq 0$, it suffices to observe that for all $s \in \mathbb{C}$ with $\Re(s) \neq 0$ and fixed y the function $H(-s)e^{s(x+iy)}$ is Schwartz on the half-line $x =\geq 0$; indeed,

$$x^n \frac{d^m}{dx^m} H(-s)e^{s(x+iy)} = H(-s)s^m x^n e^{s(x+iy)}.$$

When $\Re(s) < 0$, the exponential dominates, and thus the function is bounded; when $\Re(s) > 0$, $H(-s) = 0$, so the function is 0. Since $w \neq 0$, we can safely divide by w without affecting boundedness.

Suppose $w = 0$. Then the only new "correction" term is the function $H(-s)(x+iy)e^{s(x+iy)}$, which again on a half-line $x \geq 0$ is in the Schwartz space. ∎

Remark. With some further work it can be shown that in Lemma 4.7 the implied constants for the Schwartz functions are uniform over compact regions in (w, σ, y)-space that avoid the two lines $\sigma = 0$ and $\sigma = \Re(w)$. However we do not need this result.

Proof of Theorem 4.4. We take $s = \sigma + it$. When $\sigma \notin \{0, \Re(w)\}$, $w \neq 0$, Theorem 2.2 gives

$$\xi_{\mathbb{Q}}(w, \sigma+it) = \frac{1}{2}(\sigma+it)(\sigma+it-w)\int_{-\infty}^{\infty} e^{it(x+iy)} g_\sigma(w, x+iy)dx,$$

where $g_\sigma(w, x+iy)$ is given in Lemma 4.7. Now $g_\sigma(w, x+iy)$ is a Schwartz function in x by Lemma 4.7, so we may integrate by parts twice, using $z = x+iy$, obtaining

$$\xi_{\mathbb{Q}}(w, \sigma+it) = \frac{1}{2}\int_{-\infty}^{\infty} e^{itz}\left(\sigma - \frac{d}{dz}\right)\left(\sigma - w - \frac{d}{dz}\right) g_\sigma(w, x+iy)dx \tag{4.6}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{(\sigma+it)z}\left(-\frac{d}{dz}\right)\left(-w - \frac{d}{dz}\right) f(w, e^{2z})dx \tag{4.7}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{(\sigma+it)(x+iy)}\gamma(w, x+iy)dx \tag{4.8}$$

$$= \frac{1}{2}e^{-ty+i\sigma y}\int_{-\infty}^{\infty} (e^{\sigma x}\gamma(w, x+iy))e^{itx}dx. \tag{4.9}$$

This integral agrees with $\xi_{\mathbb{Q}}(w, s)$ off the lines $\sigma = 0, \Re(w)$. Since the integrand is uniformly Schwartz by Lemma 4.6, this integral gives an analytic function of w and s, and must therefore agree with $\xi_{\mathbb{Q}}(w, s)$ everywhere. ■

Proof of Theorem 4.3. View the integral representation of $\xi_{\mathbb{Q}}(w, \sigma - it)$ in Theorem 4.4 as a Fourier transform, with $s = \sigma + it$ with fixed $\sigma \in \mathbb{R}$. Since the Fourier transform maps Schwartz space $\mathcal{S}(\mathbb{R})$ to itself, it follows that for fixed $-\frac{\pi}{4} < y < \frac{\pi}{4}$,

$$e^{-yt}\xi_{\mathbb{Q}}(w, \sigma - it) \qquad -\infty < t < \infty,$$

belongs to $\mathcal{S}(\mathbb{R})$. The uniformity of the Schwartz constants on compact subsets Ω of (w, σ, y)-space is inherited from the corresponding uniformity property in Theorem 4.4. ■

We conclude this section with another consequence of the Fourier-Laplace integral representation of $Z_{\mathbb{Q}}(w, s)$ by (uniform) Schwartz functions.

Lemma 4.8. *Let $Q(T) \in \mathbb{C}[T]$ be any polynomial. Then for any $s = \sigma + it$ with $\sigma \notin \{0, \Re(w)\}$ and $z = x+iy$ with $-\frac{\pi}{4} < y < \frac{\pi}{4}$,*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \left(Q(\sigma+it)Z_{\mathbb{Q}}(w, \sigma+it)\right)e^{-(\sigma+it)z}dt = Q(-\frac{d}{dz})(\theta(e^{2z})^w - H(\sigma) - H(w-\sigma)e^{-wz})|_{z=x+iy}.$$

Here the integrand is a Schwartz function of t.

Proof. Using Lemma 4.7, the integral is a Fourier transform with integrand in $\mathcal{S}(\mathbb{R})$, viewing σ as fixed. The case $Q(z) \equiv 1$ follows from Theorem 2.2 by taking the Fourier transform, since the right side of that formula can be viewed as an inverse Fourier transform. Now use the fact that the Fourier transform leaves $\mathcal{S}(\mathbb{R})$ invariant and transforms multiplication by s to differentiation, and apply $Q(-\frac{d}{dz})$ to both sides of the identity with polynomial $Q(z) \equiv 1$. ■

5. Case $w = 0$

We evaluate the entire function $\xi_{\mathbb{Q}}(w, s)$ in the plane $w = 0$.

Theorem 5.1. *The entire function $\xi_{\mathbb{Q}}(w, s)$ of two complex variables has*

$$\xi_{\mathbb{Q}}(0, s) = -\frac{s^2}{8}(1 - 2^{1+\frac{s}{2}})(1 - 2^{1-\frac{s}{2}})\hat{\zeta}(\frac{s}{2})\hat{\zeta}(\frac{-s}{2}), \tag{5.1}$$

in which $\hat{\zeta}(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$.

Remarks. (1) It is evident from (5.1) that $\xi_{\mathbb{Q}}(0,s)$ satisfies the functional equation

$$\xi_{\mathbb{Q}}(0,s) = \xi_{\mathbb{Q}}(0,-s).$$

(2) Comparing the formula $\xi_{\mathbb{Q}}(0,s) = \frac{s^2}{2}\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\tilde{D}_0(\frac{s}{2})$ with Theorem 5.1, and using the functional equation for $\zeta(s)$ leads to

$$\tilde{D}_0(\frac{s}{2}) = \frac{1}{4}(1-2^{1+\frac{s}{2}})(1-2^{1-\frac{s}{2}})\zeta(\frac{s}{2})\zeta(1+\frac{s}{2}).$$

It is evident that this Dirichlet series has an Euler product, already stated in §3.4.

The proof of Theorem 5.1 depends on the Jacobi triple product formula, which is used in evaluating the Fourier coefficients of $\log\theta(t)$. We state this as a preliminary lemma.

Lemma 5.2. *The coefficients $c_m^{'}$ of*

$$\log\theta(t) = \sum_{m=1}^{\infty} c_m^{'} e^{-\pi m t}$$

are given by

$$c_m^{'} = 2\sigma_{-1}(m) - 5\sigma_{-1}(\frac{m}{2}) + 2\sigma_{-1}(\frac{m}{4}), \tag{5.2}$$

where

$$\sigma_{-1}(m) = \begin{cases} \sum_{d|m}\frac{1}{d} & m\in\mathbb{Z}_{>0} \\ 0 & \textit{otherwise.}\end{cases}$$

Proof. We use the eta product

$$\tilde{\eta}(t) := \prod_{k=1}^{\infty}(1-e^{-k\pi t}).$$

By the Jacobi triple product formula, we have

$$\begin{aligned}\theta(t) &= \prod_{k=1}^{\infty}(1+e^{-(2k-1)\pi t})^2(1-q^{-2k\pi t}) \\ &= \prod_{k=1}^{\infty}(1-e^{-k\pi t})^{-2}\prod_{k=1}^{\infty}(1-e^{-2k\pi t})^{5}\prod_{k=1}^{\infty}(1-e^{-4k\pi t})^{-2} \\ &= \tilde{\eta}(t)^{-2}\tilde{\eta}(2t)^5\tilde{\eta}(4t)^{-2}. \end{aligned} \tag{5.3}$$

If we define κ_m by

$$\log\tilde{\eta}(t) := \sum_{m=1}^{\infty}\kappa_m e^{-m\pi t},$$

then we have

$$c_m^{'} = -2\kappa_m + 5\kappa_{m/2} - 2\kappa_{m/4}, \tag{5.4}$$

with the convention that $\kappa_m = 0$ if $m \notin \mathbb{Z}$. Now,

$$\begin{aligned}\log \tilde{\eta}(t) &= \sum_{k=1}^{\infty} \log(1 - e^{-k\pi t}) \\ &= -\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{m} e^{-mk\pi t} \\ &= -\sum_{m=1}^{\infty} (\sum_{d|n} \frac{1}{d}) e^{-n\pi t}, \end{aligned} \tag{5.5}$$

so that $\kappa_m = -\sigma_{-1}(m)$ as required. ■

Proof of Theorem 5.1. Since $\xi_{\mathbb{Q}}(w, s)$ is an entire function of two variables we have, for positive real $w = u$,

$$\xi_{\mathbb{Q}}(u, s) = \lim_{u \to 0^+} \frac{s(s-u)}{2u} Z_{\mathbb{Q}}(u, s).$$

Fix $\Re(s) > 0$; for $\Re(s) > u$, we have

$$\xi_{\mathbb{Q}}(u, s) = \frac{1}{2} s(s-u) \int_0^{\infty} \frac{\theta(t^2)^u - 1}{u} t^s \frac{dt}{t} = \frac{1}{2} s(s-u) \int_0^{\infty} \frac{e^{u \log \theta(t^2)} - 1}{u} t^s \frac{dt}{t}$$

Suppose $\Re(s) > 0$. Letting $u \to 0$, and using $\lim_{u \to 0^+} \frac{e^{\alpha u} - 1}{u} = \alpha$, we eventually have $\Re(s) > u$ and so we legitimately obtain

$$\xi_{\mathbb{Q}}(0, s) = \frac{s^2}{2} \int_0^{\infty} \log(\theta(t^2)) t^s \frac{dt}{t}.$$

Now expand $\log \theta(t^2)$ in Fourier series and integrate term-by-term to obtain

$$\xi_{\mathbb{Q}}(0, s) = \frac{s^2}{4} \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) (\sum_{m=1}^{\infty} c'_m m^{-\frac{s}{2}})$$

and this converges for $\Re(s) > 2$ since $c'_m = O(\log m)$ by Lemma 5.2. Setting

$$K(s) = \sum_{m=1}^{\infty} \kappa_m m^{-s},$$

and then using (5.4) gives

$$\xi_{\mathbb{Q}}(0, s) = -\frac{s^2}{4} \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) (2 - 5 \cdot 2^{-\frac{s}{2}} + 2 \cdot 4^{-\frac{s}{2}}) K(\frac{s}{2}).$$

However we have

$$\begin{aligned} K(s) &= -\sum_{m=1}^{\infty} (\sum_{d|m} \frac{1}{d}) m^{-s} \\ &= -\prod_p \sum_{k=0}^{\infty} \frac{1 - p^{-k-1}}{1 - p^{-1}} p^{-sk} \\ &= -\prod_p (1 - p^{-s})^{-1} (1 - p^{-s-1})^{-1} \\ &= -\zeta(s)\zeta(1+s), \end{aligned} \tag{5.6}$$

valid whenever $\Re(s) > 1$ (since $\kappa_m = O(\log m)$). We thus conclude that for $\Re(s) > 2$

$$\xi_{\mathbb{Q}}(0,s) = \frac{s^2}{4}(2 - 5 \cdot 2^{-s/2} + 2 \cdot 4^{-s/2})\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(\frac{s}{2})\zeta(1+\frac{s}{2})$$

Using the duplication formula $\Gamma(\frac{s}{2}) = \frac{1}{\sqrt{2\pi}}2^{\frac{s}{2}-\frac{1}{2}}\Gamma(\frac{s}{4})\Gamma(\frac{s}{4}+\frac{1}{2})$ and the functional equation for the zeta function, we obtain for $\Re(s) > 2$ that

$$\xi_{\mathbb{Q}}(0,s) = -\frac{s^2}{8}(1 - 2^{1+s/2})(1 - 2^{1-s/2})\hat{\zeta}(s/2)\hat{\zeta}(-s/2).$$

Since both sides are analytic in s, the formula is valid for all $s \in \mathbb{C}$. ∎

6. Location of Zeros: w Positive Real

In this section we suppose $w = u > 0$ is fixed, and study the zeros of $\xi_{\mathbb{Q}}(u,s)$. We will first show that the zeros of $f_u(s)$ are localized in a vertical strip centered on the "critical line" $\mathrm{Re} s = \frac{u}{2}$, whose width depends on u, and then we shall derive an estimate for the number of zeros with imaginary part of height at most T.

6.1. Vertical Strip Bound

We show for real $w = u > 0$ that the zeros of $\xi_{\mathbb{Q}}(u,s)$ are confined to a vertical strip of width $u + 16$. Theorem 5.1 implies that this bound is valid for $u = 0$ as well.

Lemma 6.1. *Let $u > 0$ be a fixed real number. Then the entire function $f_u(s) := \xi_{\mathbb{Q}}(u,s) := \frac{1}{2}\frac{s(s-u)}{u}Z_{\mathbb{Q}}(u,s)$ has all its zeros in the vertical strip*

$$|\Re(s) - \frac{u}{2}| < \frac{u}{2} + 8. \tag{6.1}$$

Proof. We have

$$f_u(s) = \frac{s(s-u)}{4u}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)D_u\left(\frac{s}{2}\right) \tag{6.2}$$

where

$$D_u(s) = \sum_{m=1}^{\infty} c_m(u)m^{-s} \tag{6.3}$$

converges absolutely for $\Re(s) > \frac{u}{2} + 1$ by Theorem 3.3(i), and meromorphically continues to $\mathbb{C}$. All zeros of $f_u(s)$ with $\Re(s) > 0$ must come from those of the Dirichlet series $D_u(\frac{s}{2}) = 0$.

The Dirichlet series $D_u(s)$ has no zeros in any half plane $\mathrm{Re}(s) > \sigma$ for any σ with

$$|c_1(u)| > \sum_{m=2}^{\infty} |c_m(u)|m^{-\sigma}\,.$$

Since $c_1(u) = 2u$, for $u > 0$ we may rewrite this as

$$1 > \frac{1}{2}\sum_{m=2}^{\infty}\frac{|c_m(u)|}{u}m^{-\sigma} \tag{6.4}$$

Now Theorem 3.3(ii) gives

$$\frac{1}{2}\sum_{m=2}^{\infty}\frac{|c_m(u)|}{u}m^{-\sigma} \le 3\sum_{m=2}^{\infty} m^{-\sigma+\frac{u}{2}+1} .$$

Choosing $\sigma = \frac{u}{2}+4$ yields

$$\frac{1}{2}\sum_{m=2}^{\infty}\frac{|c_m(u)|}{u}m^{-\sigma} \le 3\sum_{m=2}^{\infty} m^{-3} = 3(\zeta(3)-1) < 1,$$

as required. Thus $D_u(\frac{s}{2})$ has no zeros in $\Re(s) > u+8$, hence $f_u(s)$ also has no zeros there. Finally the functional equation $f_u(s) = f_u(u-s)$ implies that $f_u(s)$ has no zeros in the region $\Re(s) < u-(u+8) = -8$. ■

Remark. The width of the strip of Lemma 6.1 is qualitatively correct, in that it must grow like $u+O(1)$ for large u and it must be of positive width, at least 4, as $u \to 0$ to accomodate the zeros of $\xi_{\mathbb{Q}}(0,s)$ given in Theorem 5.1.

6.2. Counting Zeros to Height T

We establish the following estimate for the number of zeros $N_u(T)$ within distance T of the real axis of $f_u(s)$, which generalizes a similar estimate for the Riemann zeta function.

Theorem 6.2. *There is an absolute constant C_0 such that, for all real $u > 0$ and $T \ge 0$,*

$$\frac{1}{2}N_u(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S_u(T), \tag{6.5}$$

in which $S_u(T)$ satisfies

$$|S_u(T)| \le C_0(u+1)\log(T+u+2). \tag{6.6}$$

The proof of this result generalizes the proof for $\zeta(s)$ in Davenport [9, Sect. 16], with some extra work to control the dependence in u in all estimates. We prove several preliminary lemmas.

We use the argument principle, and let $\Delta_L \arg(g(s))$ denote the change in argument θ in a function $g(s) = R(s)e^{i\theta(s)}$ along a contour L on which $g(s)$ never vanishes. For positive real u, the zeros of $f_u(s)$ are those of the analytic continuation of the Dirichlet series $D_u(\frac{s}{2})$ given in (6.3), possibly excluding zeros of $D_u(\frac{s}{2})$ at negative even integers.

We consider the rectangular contour R, oriented counterclockwise, with corners at $\sigma_0 \pm iT$ and $u-(\sigma_0 \pm iT)$, where

$$\sigma_0 = u+10,$$

and T is chosen to avoid any zeros of $f_u(s)$. (See Figure 5.1) We will mainly use the quarter-contour L consisting of a vertical line V from $z_0 = \sigma_0$ to $z_1 = \sigma_0 + iT$, followed by a horizontal line H from z_1 to $z_2 = \frac{u}{2}+iT$.

Lemma 6.3. *For real $u > 0$, and $T \ge 2$,*

$$\frac{1}{2}N_u(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S_u(T) \tag{6.7}$$

with

$$S_u(T) = \Delta_L \arg D_u\left(\frac{s}{2}\right) + O\left(\frac{u}{T}\right) . \tag{6.8}$$

Proof. We have

$$2\pi N_u(T) = \Delta_R \arg(f_u(s))$$

on the rectangular contour R , oriented counterclockwise, which has its corners at $\sigma_0 \pm iT$ and $\frac{u}{2} \pm \sigma_0 \pm iT$, as given above.

The functional equation $f_u(s) = f_u(u-s)$ and the symmetry $f_u(r) = \overline{f_u(\bar{s})}$ imply that

$$\frac{\pi}{2} N_u(T) = \arg_L(f_u(s))\,, \tag{6.9}$$

on the quarter-contour L, with each other quarter-contour of R contributing the same amount. Now

$$u f_u(s) = \frac{1}{2}(s-u)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}+1\right) D_u\left(\frac{s}{2}\right)\,, \tag{6.10}$$

so we obtain

$$\frac{\pi}{2} N_u(T) = \Delta_L \arg\left(\pi^{-\frac{s}{2}}\right) + \Delta_L \arg\left(\frac{s}{2}\Gamma(\frac{s}{2})\right) + \Delta_L \arg\left(s-u\right) + \Delta_L \arg\left(D_u(\frac{s}{2})\right)\,. \tag{6.11}$$

The first three terms on the right contribute

$$\begin{aligned} \Delta_L \arg\left(\pi^{-\frac{s}{2}}\right) &= \Delta_L\left(-\frac{1}{2} t \log \pi\right) = -\frac{T}{2}\log \pi\,, \\ \Delta_L \arg\left(s-u\right) &= \arg\left(iT - \frac{u}{2}\right) = \frac{\pi}{2} + O\left(\frac{u}{T}\right)\,, \\ \Delta_L \arg \Gamma\left(\frac{s}{2}+1\right) &= \frac{T}{2}\log\left(\frac{T}{2}\right) - \frac{T}{2} + \frac{3\pi}{8} + O\left(\frac{1}{T}\right) \end{aligned} \tag{6.12}$$

where Stirling's formula is used for the last estimate. This yields (6.8). ■

Our object will be to estimate $\Delta_L \arg D_u\left(\frac{s}{2}\right)$ using the formula

$$\Delta_L \arg\left(D_u(\frac{s}{2})\right) = -\int_L \Im\left(\frac{1}{2}\frac{D_u'(\frac{s}{2})}{D_u(\frac{s}{2})}\right) ds\,,$$

starting from the endpoint $s = \sigma_0$ of L , where the next lemma shows $D_u\left(\frac{\sigma_0}{2}\right)$ is real and positive and $-\frac{1}{2}\frac{D_u'(\frac{\sigma_0}{2})}{D_u(\frac{\sigma_0}{2})}$ is real and positive. In the integral we analytically continue $\frac{D_u'(\frac{s}{2})}{D_u(\frac{s}{2})}$ along L, and we choose T so that the contour L encounters no zero of $D_u\left(\frac{s}{u}\right)$. We need information on the zeros $\mathcal{Z}_u$ of $D_u(s)$ obtained from the Hadamard product.

Lemma 6.4. *Let $u > 0$ be real.*

(i) For $s = \sigma + it \notin \mathcal{Z}_u \cup \{-2,-4,-6,\dots\}$ there holds

$$\sum_{\rho\in\mathcal{Z}_u} \frac{\sigma - \Re(\rho)}{|s-\rho|^2} = \frac{1}{2}\Re\left(\frac{D_u'(\frac{s}{2})}{D_u(\frac{s}{2})}\right) + \frac{1}{2}\Re\left(\frac{\Gamma'(\frac{s}{2}+1)}{\Gamma(\frac{s}{2}+1)}\right) + \frac{\sigma-u}{|s-u|^2} - \frac{1}{2}\log\pi\,. \tag{6.13}$$

(ii) For $\Re(s) > u+8$,

$$\Re\left(D_u\left(\frac{s}{2}\right)\right) > 0\,, \tag{6.14}$$

and there is an absolute constant A_0 independent of u such that

$$\left|\frac{D_u'\left(\frac{s}{2}\right)}{D_u\left(\frac{s}{2}\right)}\right| \le A_0. \tag{6.15}$$

Proof. (i). We use the Hadamard product expansion

$$u f_u(s) = \frac{1}{2} \frac{s(u-s)}{u} Z(u,s) = e^{A(u)+B(u)s} \prod_{\rho \in \mathcal{Z}_u} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}} \tag{6.16}$$

where $\mathcal{Z}_u$ is the set of zeros of $f_u(s)$ counted with multiplicity. This formula is valid because $u f_u(s)$ is an entire function of order at most one by the growth estimate of Theorem 4.1. Note that $0 \notin \mathcal{Z}_u(s)$ because for fixed u the function $Z(u,s)$ has a simple pole with residue 1 at $s = 0$. The derivation of [9, pp. 82–84] yields the formula

$$B(u) = \Re(B(u)) = -\sum_{\rho \in \mathcal{Z}_u}{}' \frac{1}{\rho} = -\sum_{\rho \in \mathcal{Z}_u} \frac{\Re(\rho)}{|\rho|^2}\,, \tag{6.17}$$

where the prime in the first sum indicates that complex conjugate zeros ρ and $\bar{\rho}$ are to be summed in pairs, and the last sum converges absolutely.

We set equal the logarithmic derivatives of (6.10) and (6.16), to obtain

$$\frac{1}{2} \frac{D_u'\left(\frac{s}{2}\right)}{D_u\left(\frac{s}{2}\right)} + \frac{1}{2} \frac{\Gamma'\left(\frac{s}{2}+1\right)}{\Gamma\left(\frac{s}{2}+1\right)} + \frac{1}{s-u} - \frac{1}{2}\log \pi = B(u) + \sum_{\rho \in \mathcal{Z}_u} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right). \tag{6.18}$$

This yields

$$-\frac{1}{2} \frac{D_u'\left(\frac{s}{2}\right)}{D_u\left(\frac{s}{2}\right)} = \frac{1}{s-u} - B(u) - \frac{1}{2}\log \pi + \frac{1}{2} \frac{\Gamma'\left(\frac{s}{2}+1\right)}{\Gamma\left(\frac{s}{2}+1\right)} - \sum_{\rho \in \mathcal{Z}_u} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right). \tag{6.19}$$

Taking real parts yields,

$$\begin{aligned} -\frac{1}{2}\Re\left(\frac{D_u'\left(\frac{s}{2}\right)}{D_u\left(\frac{s}{2}\right)}\right) \quad &= \quad \frac{\sigma-u}{|s-u|^2} - \Re(B(u)) - \frac{1}{2}\log\pi \\ &\quad + \frac{1}{2}\Re\left(\frac{\Gamma'\left(\frac{s}{2}+1\right)}{\Gamma'\left(\frac{s}{2}+1\right)}\right) - \sum_{\rho \in \mathcal{Z}_u} \left(\frac{\sigma - \Re(\rho)}{|s-\rho|^2} + \frac{\Re(\rho)}{|\rho|^2}\right). \end{aligned} \tag{6.20}$$

Now (6.20) holds in the entire plane by analytic continuation, since the functions are single-valued. Applying the formula for $B(u)$ simplifies (6.20) to (6.13), proving (i).

(ii) Suppose $\Re(s) > u + 8$. Then the formula (6.10) for $D_u(s)$ and Theorem 3.3(ii) give

$$\begin{aligned} \left| D_u\left(\frac{s}{2}\right) - 2u \right| \quad &\leq \quad \sum_{m=2}^{\infty} |c_m(u)| m^{-\sigma/2} \\ &\leq \quad 6u \sum_{m=1}^{\infty} m^{\frac{u+1-\sigma}{2}} \leq 6u(\zeta(3) - 1) \\ &\leq \quad \frac{3}{2} u\,. \end{aligned} \tag{6.21}$$

This implies (6.14). Next, applying Theorem 3.3(ii) again,

$$\begin{aligned}
\left|\frac{D_u'\left(\frac{s}{2}\right)}{D_u\left(\frac{s}{2}\right)}\right| &\leq \frac{1}{\left(2-\frac{3}{2}\right)u}\left|D_u'\left(\frac{s}{2}\right)\right| \\
&\leq \frac{2}{u}\sum_{m=2}^{\infty}|c_m(u)|\frac{\log m}{2}m^{-\sigma/2} \\
&\leq 24\sum_{m=2}^{\infty}(\log m)m^{\frac{u+1-\sigma}{2}} \\
&\leq 24\sum_{m=2}^{\infty}(\log m)m^{-3} = O(1)\,,
\end{aligned}$$

which gives (6.15). ∎

Lemma 6.5. *(i) There is an absolute constant A_1, such that for all $u > 0$, and all real T,*

$$\sum_{\rho=\beta+i\gamma\in\mathcal{Z}_u}\frac{u+2}{4(u+9)^2+(T-\gamma)^2} \leq A_1\log(|T|+u+2)\,. \tag{6.22}$$

(ii) There is an absolute constant A_2 such that for all $u > 0$ and all T, the number of zeros $\rho = \beta + i\gamma \in \mathcal{Z}_u$ with

$$|\gamma - T| < u + 9, \tag{6.23}$$

counting multiplicity, is at most

$$A_2(u+9)\log(|T|+u+2). \tag{6.24}$$

Proof. (i). Choose $s = \sigma_0 + iT$ with

$$\sigma_0 = 2u + 10\,, \tag{6.25}$$

so $\frac{\sigma_0-u}{|s-u|^2} \leq \frac{u+10}{|\sigma_0-u|^2} \leq \frac{1}{10}$. Now apply (6.13) and the bound (6.15) to obtain

$$\sum_{\rho\in\mathcal{Z}_u}\frac{\sigma_0-\Re\rho}{|s-\rho|^2} = \frac{1}{2}\Re\left(\frac{\Gamma'\left(\frac{s}{2}+1\right)}{\Gamma\left(\frac{s}{2}+1\right)}\right) + O(1)\,. \tag{6.26}$$

We recall the formula

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s + O\left(\frac{1}{|s|}\right) \tag{6.27}$$

valid for $-\pi + \delta < |\arg s| < \pi - \delta$ for any fixed δ. (We choose $\delta = \frac{\pi}{2}$.) Now (6.25) gives

$$\begin{aligned}
\Re\left(\frac{\Gamma'\left(\frac{s}{2}+1\right)}{\Gamma\left(\frac{s}{2}+1\right)}\right) &= \Re\left(\log\frac{s}{2}+1\right) + O(1) \\
&= \log\left|\frac{s}{2}+1\right| + O(1) \\
&= \log(|\sigma_0| + |T|)\,. \qquad (6.28)
\end{aligned}$$

Thus (6.26) becomes

$$\sum_{\rho\in\mathcal{Z}_u} \frac{\sigma_0-\beta}{(\sigma_0-\beta)^2+(T-\gamma)^2} = O(\log(|T|+2u+8))\ . \tag{6.29}$$

The bound of Lemma 6.1 gives

$$u+2 \le \sigma_0 - \Re(\rho) \le 2(u+9)\ .$$

Thus

$$\frac{u+2}{4(u+9)^2+(T-\gamma)^2} \le \frac{\sigma_0-\beta}{(\sigma_0-\beta)^2+(T-\gamma)^2}\ ,$$

and (6.22) follows.

(ii). Let S_T denote the set of zeros in $\mathcal{Z}_u$ satisfying $|\gamma - T| < u+9$. Then, since $u \ge 0$,

$$\sum_{\rho\in S_T} \frac{u+2}{4(u+9)^2+|\gamma-T|^2} \ge \frac{2}{45}\left(\frac{|S_T|}{u+9}\right)\ .$$

Combining this with (6.22) implies (6.24) with $A_2 = \frac{45}{2}A_1$. ■

Lemma 6.6. *There is an absolute constant A_3, such that for $u > 0$ and $s = \sigma + iT$ with $s \notin \mathcal{Z}_u$ in the region*

$$\left|\sigma - \frac{u}{2}\right| \le \frac{3u}{2} + 10\ , \tag{6.30}$$

there holds, for all $|T| \ge |\sigma| + 1$,

$$\left|\frac{1}{2}\frac{D_u'\left(\frac{s}{2}\right)}{D_u\left(\frac{s}{2}\right)} - \sum_{\substack{\rho\in\mathcal{Z}_u\\ |\gamma-T|\le u+9}} \frac{1}{s-\rho}\right| \le A_3 \log(|T|+u+2). \tag{6.31}$$

Proof. Set $\sigma_0 = 2u+10$, and $s_0 = \sigma_0 + iT$. Then, differencing (6.19) at s and s_0, we obtain

$$\begin{aligned}\frac{1}{2}\frac{D_u'\left(\frac{s}{2}\right)}{D_u\left(\frac{s}{2}\right)} \quad - \quad \frac{1}{2}\frac{D_u'\left(\frac{s_0}{2}\right)}{D_u'\left(\frac{s_0}{2}\right)} &= \left(-\frac{1}{s-u}+\frac{1}{s_0-u}\right)\\ &\quad + \sum_{\rho\in\mathcal{Z}_u}\left\{\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right)-\left(\frac{1}{s_0-\rho}+\frac{1}{\rho}\right)\right\}\\ &\quad - \quad \frac{1}{2}\left(\frac{\Gamma'\left(\frac{s}{2}+1\right)}{\Gamma'\left(\frac{s}{2}+1\right)} - \frac{\Gamma'\left(\frac{s_0}{2}+1\right)}{\Gamma'\left(\frac{s_0}{2}+1\right)}\right)\ .\end{aligned}$$

The bound $|T| \ge |\sigma|+1$ ensures that $s = \sigma + iT$ has $-\pi+\delta < \arg(s) < \pi+\delta$ for $\delta = \frac{\pi}{4}$, hence the bounds (6.28) applies to give

$$\frac{1}{2}\frac{D_u'\left(\frac{s}{2}\right)}{D_u\left(\frac{s}{2}\right)} = \sum_{\rho\in\mathcal{Z}_u}\left(\frac{1}{s-\rho}-\frac{1}{s_0-\rho}\right) + O(1) + O(\log(|T|+u+2))\,. \tag{6.32}$$

Now

$$\left|\frac{1}{s-\rho}-\frac{1}{s_0-\rho}\right| = \frac{|s-s_0|}{|s-\rho||s_0-\rho|} \quad \leq \quad \frac{3u+20}{|s-\rho||s_0-\rho|}$$
$$\leq \quad \frac{4(u+5)}{|\gamma-T|^2}\,. \tag{6.33}$$

For those zeros with $|\gamma - T| > u+9$, Lemma 6.5(i) gives the bound

$$\sum_{\substack{\rho\in\mathcal{Z}_u \\ |\gamma-T>u+9}} \frac{4(u+5)}{(\gamma-T)^2} \quad \leq \quad 60\sum_{\rho\in\mathcal{Z}_u}\frac{u+2}{4(u+9)^2+|\gamma-T|^2}$$
$$\leq \quad 60A_1(\log(|T|+u+2)).$$

If $|\gamma-T|\leq u+9$, then by Lemma 6.5(ii) the number of such zeros is $O((u+1)\log(|T|+u+2))$ and for each one

$$\left|\frac{1}{s_0-\rho}\right| = O\left(\frac{1}{|\sigma_0-\beta|}\right) = O\left(\frac{1}{u+9}\right)\,.$$

So their total contribution is $O(\log|T|+u^*))$ in (6.32). Substituting these bounds in (6.32) yields

$$\left|\frac{1}{2}\frac{D_u'\left(\frac{s}{2}\right)}{D_u\left(\frac{s}{2}\right)} - \sum_{\substack{\rho\in\mathcal{Z}_u \\ |\rho-T|<u+9}}\frac{1}{s-\rho}\right| = O(\log(|T|+u+2))\,,$$

as required. ∎

Proof of Theorem 6.2. Set $\sigma_0 = 2u+10$. Recall that the quarter- contour L consists of the vertical segment V from σ_0 to σ_0+iT, and the horizontal segment H from σ_0+iT to $\frac{u}{2}+iT$, so that

$$\pi S_u(T) = \Delta_V \arg D_u\left(\frac{s}{2}\right) + \Delta_H \arg D_u\left(\frac{s}{2}\right) + O\left(\frac{u}{T}\right) \tag{6.34}$$

by (6.12). We have

$$\left|\Delta_V \arg D_u\left(\frac{s}{2}\right)\right| \leq \pi,$$

using Lemma 6.4(ii). Now

$$\Delta_H \arg D_u\left(\frac{s}{2}\right) = -\int_{\frac{1}{2}+iT}^{\sigma_0+iT} \Im\left(\frac{1}{2}\frac{D_u'\left(\frac{s}{2}\right)}{D_u\left(\frac{s}{2}\right)}\right) ds\,.$$

and to estimate this we apply Lemma 6.6. For each zero ρ we have

$$\int_{\frac{1}{2}+iT}^{\sigma_0+iT}\Im\left(\frac{1}{s-\rho}\right) ds = \Delta_H \arg(s-\rho),$$

which contributes at most π. Now Lemma 6.5(ii) gives that there are at most $O((u+1)\log(|T|+u+2))$ such zeros in the sum (6.31), so their total contribution to the argument is at most $O((u+1)\log(|T|+u+2))$. The error term in (6.31) integrated over H contributes at most a further $O((u+1)\log(|T|+u+2))$ to the argument, since H is a path of length at most $\frac{3u}{2}+10$. Combining these estimates in (6.34) yields the bound (6.6), completing the proof. ∎

Remark. The zero-counting estimate of Theorem 6.2 for $u > 0$ is easily checked to remains valid for $u = 0$ by virtue of the explicit formula for $f_0(s) = \xi_{\mathbb{Q}}(0, s)$ in Theorem 5.1. Note that the extra zeros provided by the terms $(1 - 2^{1+\frac{s}{2}})(1 - 2^{1-\frac{s}{2}})$ are needed to make the main term (6.5) valid.

6.3. Movement of Zeros

It is interesting to examine the behavior of the zero set of $\xi_{\mathbb{Q}}(u, s)$ for nonnegative real u, as u is varied. As above, consider variation $1 \le u \le 2$, or, more generally, over a fixed bounded range of u. For that range of u, Theorem 6.2 asserts that the general density of zeros to height T remains almost constant, with a variation of $O(\log(T + 2))$. Since the number of zeros in a unit interval at this height is of the same order, it suggests that every zero can move vertically a distance of at most $O(1)$, while the horizontal movement is certainly restricted to distance $O(1)$ by Lemma 6.1. Therefore it would appear that varying $1 \le u \le U$, there is a constant C_U depending on U such that every zero moves at most the bounded amount C_U, *independent of the height T of this zero* in the critical strip. We cannot assert this rigorously, however, because we have not ruled out the possibility of zeros going off the line in pairs and then hop-scotching around other zeros remaining on the line.

Theorem 6.2 for $u = 1$ and $u = 2$ shows that the zero-counting functions of $\zeta(s)$ and $\zeta_{\mathbb{Q}(i)}(\frac{s}{2})$ have extemely similar asymptotics. In Table 1 we compare the first 25 such zeros. Since $\zeta_{\mathbb{Q}(i)}(s) = \zeta(s)L(s, \chi_{-4})$, we have indicated the zeros of $\zeta_{\mathbb{Q}(i)}(\frac{s}{2})$ associated to $\zeta(\frac{s}{2})$ with an asterisk, and the remainder come from $L(\frac{s}{2}, \chi_{-4})$. Note that the appearance of every zeta zero on both sides of this table shows that zeta zeros have a kind of self-similar structure by powers of two. However this is only approximately true, because there is no precise correspondence of zeta zeros due to the phenomonon of zeros coalescing as u varies, as indicated in Lemma 8.1 in §8.

7. Location of Zeros: w Negative Real

In this section we suppose $w = -u$ is negative real ($u > 0$) and fixed. We will not find zeros, but will instead specify places in the s-plane where the the function $\xi_{\mathbb{Q}}(u, s)$ has no zeros. An interesting extra structure underlying certain properties of the two-variable zeta function $Z_{\mathbb{Q}}(w, s)$ for negative real w is a holomorphic convolution semigroup of complex-valued measures described in §7.2. On the critical line $\Re(s) = -\frac{u}{2}$, we will show these are real-valued positive measures, normalizable to be probability measures, in §7.1.

7.1. Positivity on Critical Line

We will establish the following result, concerning the absence of zeros on the "critical line" $\Re(s) = -\frac{u}{2}$ ($u > 0$), which will be deduced from Theorem 7.4 below.

Theorem 7.1. (*Positivity Property*) *For real $w = -u$ with $u > 0$ and all real t,*

$$Z_{\mathbb{Q}}\left(-u, -\frac{u}{2} + it\right) > 0 \ . \tag{7.1}$$

Recall that the functional equation on the "critical line" $s = -\frac{u}{2} + it$ implies that

$$Z_{\mathbb{Q}}\left(-u, -\frac{u}{2} + it\right) = Z_{\mathbb{Q}}\left(-u, -\frac{u}{2} - it\right) = \overline{Z_{\mathbb{Q}}\left(-u, -\frac{u}{2} + it\right)} \ ,$$

	$\zeta(s)$	$\zeta_{\mathbb{Q}(i)}(s/2)$	
1	14.13	12.04	
2	21.02	20.48	
3	25.01	25.96	
4	30.42	28.26	$*$
5	32.94	32.68	
6	37.58	36.58	
7	40.91	42.04	$*$
8	43.32	42.90	
9	48.00	46.54	
10	49.77	50.02	$*$
11	52.77	51.44	
12	56.44	56.78	
13	59.34	59.30	
14	60.83	60.85	$*$
15	65.11	65.18	
16	67.07	65.87	$*$
17	69.54	68.38	
18	72.06	72.28	
19	75.70	75.16	$*$
20	77.14	77.02	
21	79.33	80.64	
22	82.91	81.82	$*$
23	84.73	83.60	
24	87.42	86.64	$*$
25	88.81	89.22	

Table 1: Imaginary part of zeros of $\zeta(s)$ and $\zeta_{\mathbb{Q}(i)}(s/2)$. ($*$ = zero of $\zeta(s/2)$.)

hence $Z_{\mathbb{Q}}(-u, -\frac{u}{2} + it)$ is real. For real $-u < 0$, the integral representation

$$Z_{\mathbb{Q}}\left(-u, -\frac{u}{2} + it\right) = \int_0^{\infty} \theta(x^2)^u x^{-\frac{u}{2}+it} \frac{dx}{x} \tag{7.2}$$

converges absolutely for all $t \in \mathbb{R}$, and it gives

$$Z_{\mathbb{Q}}\left(-u, -\frac{u}{2}\right) > 0 \ , \tag{7.3}$$

since the integrand is positive. Thus the assertion that $Z_{\mathbb{Q}}(-u, -\frac{u}{2} + it) \neq 0$ for all real t is equivalent to the positivity condition (7.1). By changing variables in the integral (7.2), with $x = e^{-r}$, we obtain

$$\begin{aligned} \frac{1}{2\pi} Z_{\mathbb{Q}}\left(-u, -\frac{u}{2} + it\right) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \theta(e^{-2r})^{-u} e^{\frac{ru}{2}} e^{-irt} dr \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{e^{r/2}}{\theta(e^{-2r})} \right)^u e^{-irt} dr. \end{aligned} \tag{7.4}$$

This implies that the function

$$P_u(x) := \frac{1}{2\pi} \theta(1)^u Z_{\mathbb{Q}}(-u, -\frac{u}{2} + ix) \tag{7.5}$$

has inverse Fourier transform

$$\check{P}_u(r) := \int_{-\infty}^{\infty} P_u(x) e^{ixr} dx = \theta(1)^u \left(\frac{e^{r/2}}{\theta(e^{-2r})} \right)^u . \tag{7.6}$$

The function $P_u(x)$ is an even function and has

$$\int_{-\infty}^{\infty} P_u(x) dx = \check{P}_u(0) = 1. \tag{7.7}$$

This equation shows that $P_u(x)dx$ is a signed measure of mass one. Theorem 7.1 asserts that for $P_u(x) > 0$ for all x holds for all $u > 0$, which would imply that $P(x)dx$ is a probability measure. In any case $\check{P}_u(r)$ is the characteristic function of the (signed) measure $P_u(x)dx$, and (7.6) shows that $\check{P}_u(r) = f(r)^u$ where

$$f(r) := \theta(1) \frac{e^{r/2}}{\theta(e^{-2r})} = \frac{\theta(1)}{\sqrt{\theta(e^{2r})\theta(e^{-2r})}}, \tag{7.8}$$

where the last expression is derived using the functional equation for the theta function. The assertion that $f(r)^u$ is a characteristic function for all real $u > 0$ is equivalent to the assertion that each $P_u(x)dx$ is an *infinitely divisible* probability measure; the collection $\{P_u(x)dx \ : u > 0\}$ then form a semigroup under convolution.

We note that the normalizing factor

$$\theta(1) = 1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2} = \frac{\pi^{\frac{1}{4}}}{\Gamma\left(\frac{3}{4}\right)} \approx 1.08643 \tag{7.9}$$

appearing in (7.5) is the invariant $\eta(\mathbb{Q})$ introduced by van der Geer and Schoof [11, p. 16]. They define the *genus of* $\mathbb{Q}$ to be the "dimension" $h^0(\kappa_{\mathbb{Q}})$ of the canonical divisor $\kappa_{\mathbb{Q}} = (1)$, which is

$$h_0(\kappa_{\mathbb{Q}}) = \log(\eta(\mathbb{Q})\sqrt{\Delta_{\mathbb{Q}}}) = \log \theta(1) \ , \tag{7.10}$$

see the appendix.

A necessary and sufficient condition for a function to be a characteristic function of an infinitely divisible probability measure was developed by Khintchine, Levy and Kolmogrov. We follow the treatment in Feller [10, p. 558–563]. A measure $M\{dy\}$ on $\mathbb{R}$ is called *canonical* if it is nonnegative, assigns finite masses to finite intervals and if both the integrals

$$M^+(x) = \int_x^\infty \frac{1}{y^2} M\{dy\}, \quad M^-(x) = \int_{-\infty}^{-x} \frac{1}{y^2} M\{dy\} \tag{7.11}$$

converge for some (and therefore all) $x > 0$.

Proposition 7.2. *A complex-valued function $f(r)$ on $\mathbb{R}$ is the characteristic function of an infinitely divisible probability measure if and only if $f(r) = \exp(\psi(r))$ with $\psi(r)$ having the form*

$$\psi(r) = ibr + \int_{-\infty}^{\infty} \frac{e^{irx} - 1 - ir \sin x}{x^2} M\{dx\} \tag{7.12}$$

for some canonical measure M and real constant b. The canonical measure M and constant b are unique.

Proof. This is shown in Feller [10, pp. 558–563]. ∎

The representation (7.12) implies that $f(0) = 1$, $f(-r) = \overline{f(r)}$ for all r and that $\log f(r)$ is well-defined, with its imaginary part determined by continuity starting from $\log f(0) = 0$.

We call the measure $M\{dx\}$ in Proposition 7.2, which may have infinite mass, the *Feller canonical measure* associated to $f(r)$. A related canonical measure is the *Khintchine canonical measure* $K\{dx\}$, given by

$$K\{dx\} = \frac{1}{1+x^2} M\{dx\}.$$

It can be an arbitrary bounded nonnegative measure, see Feller [10, pp. 564-5]. Biane, Pitman and Yor [4, p. 9] consider the *Levy-Khintchine canonical measure* $\nu\{dx\}$, which is defined for infinitely divisible distributions supported on $[0, \infty)$, and is related to the corresponding Feller canonical measure by

$$\nu\{dx\} = \frac{1}{x^2} M\{dx\}.$$

Note that the Feller canonical measure $M\{dx\}$ given in Theorem 1.2 has support on the whole real line, so has no associated Levy-Khintchine canonical measure.

We will use a variant of this result which characterizes infinitely divisible distributions with finite second moment.

Proposition 7.3. *A complex-valued function $f(r)$ on $\mathbb{R}$ is the characteristic function of an infinitely divisible probability distribution having a finite second moment if (and only if) $f(r)$ is a C^2-function, $f(0) = 1$, $f(-r) = \overline{f(r)}$, for all r, $f(r) \neq 0$, and*

$$g(r) := \frac{f''(r)f(r) - f'(r)^2}{f(r)^2} = \frac{d^2}{dr^2}(\log f(r)) \tag{7.13}$$

has $g(0) < 0$ and $-g(r)$ is the characteristic function of a positive measure $M\{dx\}$ of finite mass,

$$-g(r) = \int_{-\infty}^{\infty} e^{irx} M\{dx\} \ . \tag{7.14}$$

If so, then $M\{dx\}$ is the associated Feller canonical measure to $f(r)$.

Proof. The "if" part of this result appears in Feller [10, p. 559 bottom]. We will not use the "only if" part of the result and omit its proof. ∎

Theorem 7.4. *The function*

$$f(r) = \theta(1) \frac{e^{r/2}}{\theta(e^{-2r})} = \frac{\theta(1)}{\sqrt{\theta(e^{-2r})\theta(e^{2r})}} = \theta(1) \frac{e^{-r/2}}{\theta(e^{2r})} \ , \tag{7.15}$$

on $\mathbb{R}$ is the characteristic function of an infinitely divisible probability measure with finite second moment. Its associated Feller canonical measure $M\{dx\}$ is equal to $M(x)dx$ with

$$M(x) = \frac{1}{8\pi} x^2 |1 - 2^{1+\frac{ix}{2}}|^2 \left| \hat{\zeta}\left(\frac{ix}{2}\right) \right|^2 . \tag{7.16}$$

in which $\hat{\zeta}(s) := \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$.

Proof. We apply the criterion of Proposition 7.3. We start from the function

$$g(r) = \frac{d^2}{dr^2} log(\theta(1) \frac{e^{-r/2}}{\theta(e^{2r})}) = -\frac{d^2}{dr^2} \log \theta(e^{2r}),$$

and must show that $g(0) < 0$ and that

$$M(x) := -\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-irx} g(r) dr$$

is a nonnegative function having finite mass. We have $g(0) \approx -1.8946 < 0$ and

$$M(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-irx} \left(\frac{d^2}{dr^2} \log \theta(e^{2r}) \right) dr = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-irx} \gamma(0, r) dr,$$

using (4.5). Now Theorem 4.4 gives, on taking $w = 0$ and $y = 0$, that for all $s \in \mathbb{C}$,

$$\xi_{\mathbb{Q}}(0, s) = \frac{1}{2} \int_{-\infty}^{\infty} e^{rs} \gamma(0, r) dr,$$

and it follows that $M(x) = \frac{1}{\pi} \xi_{\mathbb{Q}}(0, -ix)$. Now Theorem 5.1, which uses the Jacobi triple product formula, gives

$$\xi_{\mathbb{Q}}(0, -ix) = \xi_{\mathbb{Q}}(0, ix) = \frac{x^2}{8} (1 - 2^{1+\frac{ix}{2}})(1 - 2^{1+\frac{-ix}{2}}) \hat{\zeta}(\frac{ix}{2}) \hat{\zeta}(\frac{-ix}{2}),$$

Using $\hat{\zeta}(\frac{-ix}{2}) = \overline{\hat{\zeta}(\frac{ix}{2})}$ we obtain

$$M(x) = \frac{1}{\pi}\xi_{\mathbb{Q}}(0, ix) = \frac{1}{8\pi}x^2|1 - 2^{1+\frac{ix}{2}}|^2|\hat{\zeta}(\frac{ix}{2})|^2.$$

This shows that $M(x)$ is nonnegative, and its strict positivity follows from the well-known result that $\zeta(s)$ is nonzero on the line $\Re(s) = 1$, using $\hat{\zeta}(ix) = \hat{\zeta}(1 - ix)$. The positive measure $M(x)dx$ has finite mass since $\xi_{\mathbb{Q}}(0, ix)$ is a Schwartz function by Theorem 4.3, so the result follows by Proposition 7.3. (The mass of $M(x)dx$ is explicitly determined in Theorem 7.9 below; numerically it is about 1.8946.) ∎

Proof of Theorem 7.1. For real $u > 0$ the function $f(r)^u$ with $f(r)$ given by (7.15) is by Theorem 7.4 the characteristic function of a probability density of a nonnegative measure. The measure $P_u(x)dx$ is positive except on a discrete set because the density $P_u(x)dx$ is an analytic function of x. Then, using the infinite divisibility property, we have

$$P_u(y) = P_{u/2} * P_{u/2}(y) = \int_{-\infty}^{\infty} P_{\frac{u}{2}}(x)P_{\frac{u}{2}}(y - x)dx > 0,$$

giving positivity for all real y. The probability density $P_u(x)dx$ is given by (7.5), and $\theta(1) > 0$, so we conclude that $Z_{\mathbb{Q}}(-u, -\frac{u}{2} + ix) > 0$ for all real x. ∎

7.2. Holomorphic Convolution Semigroup

In §7.1 we showed that for $u > 0$ the family of probability measures $\{\rho_u := P_u(x)dx : \ u > 0\}$ having the density functions

$$P_u(x) = \frac{1}{2\pi}\theta(1)^u Z_{\mathbb{Q}}\left(-u, \frac{-u}{2} + ix\right) \ ,$$

form a semigroup under convolution, i.e. $\rho_{u_1} * \rho_{u_2} = \rho_{u_1+u_2}$. We can extend this to a convolution semigroup of complex-valued measures on the real line, indexed by two real parameters (u, v) with $u > 0$ and $|v| < u$, a region which forms an open cone in $\mathbb{R}^2$ closed under addition. Given such (u, v) we define a complex-valued measure $\rho_{u,v}(x)dx$ on the real line by

$$\rho_{u,v}(x) = \frac{1}{2\pi}\theta(1)^u Z_{\mathbb{Q}}(-u, -\frac{u + v}{2} + ix) \qquad \text{for} \qquad -\infty < x < -\infty. \tag{7.17}$$

In terms of the function $Z_{\mathbb{Q}}(w, s)$ these values occupy the real-codimension one cone

$$\mathcal{C}^- = \{(w, s) \ : \ w = u \in \mathbb{R} \ \text{ and } \ u < \Re(s) < 0\}, \tag{7.18}$$

which is contained in the absolute convergence region $\mathcal{C}$ of the integral representation (1.3).

Lemma 7.5. *For all $u > 0$ and real $|v| < u$, $\rho_{u,v}(x)$ is a complex-valued measure with*

$$\int_{-\infty}^{\infty} \rho_{u,v}(x)dx = 1.$$

The measures $\{\rho_{u,v} : u > 0, \ |v| < u\}$ form a convolution semigroup. That is, for $u_1, u_2 > 0$, $|v_1| < u_1, |v_2| < u_2$ and $x \in \mathbb{R}$, we have:

$$\int_{-\infty}^{\infty} \rho_{u_1,v_1}(y)\rho_{u_2,v_2}(x - y)dy = \rho_{u_1+u_2,v_1+v_2}(x).$$

In particular

$$\int_{-\infty}^{\infty} \rho_{u,v}(x)^2 dx = \rho_{2u,2v}(0).$$

Proof. The first formula follows from the formula

$$Z_{\mathbb{Q}}(-u,-\frac{u+v}{2}+ix)=\check{\mathcal{F}}\left((\frac{e^r}{\theta(e^{-2r})})^u e^{vr}\right)(-\frac{x}{\pi}) \tag{7.19}$$

which generalizes (7.4). Indeed the functions

$$f_{u,v}(r)=\left(\frac{e^{\frac{r}{2}}}{\theta(e^{-2r})}\right)^u e^{\frac{vr}{2}} \qquad \text{for} \qquad -\infty < r < -\infty$$

form a semigroup under multiplication, i.e.

$$f_{u_1,v_1}(r)f_{u_2,v_2}(r)=f_{u_1+u_2,v_1+v_2}(r)$$

and the inverse Fourier transform relation (7.19) implies that the measures $\rho_{u,v}(x)dx$ form a semigroup under convolution, i.e.

$$P_{u_1,v_1}(x)dx * P_{u_2,v_2}(x)dx = P_{u_1+u_2,v_1+v_2}(x)dx,$$

as required. ∎

This semigroup is *holomorphic* in the sense that the density functions $\rho_{u,v}(x)$ are holomorphic functions of $s=-\frac{u+v}{2}+ix$ in the cone where they are defined. Below we compute the moments of the distributions $\rho_{u,v}(x)dx$ using Lemma 4.8.

Before doing that, we derive a formula for the logarithmic derivatives of the theta function $\theta(t)$ at $t=1$. Set

$$R_k := \frac{d^k}{dt^k}\log\theta(t)|_{t=1}. \tag{7.20}$$

Theorem 7.6. *For each $k\geq 1$,*

$$R_k \in \mathbb{Q}[\psi_2], \tag{7.21}$$

where

$$\psi_2 := \pi\theta(1)^4 = \frac{\pi^2}{\Gamma(3/4)^4}.$$

It is given by an even polynomial of degree at most k in ψ_2,

Remark. In particular, we have

$$\begin{aligned} R_1 &= -\frac{1}{4} \\ R_2 &= \frac{1}{8}+\frac{1}{32}\psi_2^2 \\ R_3 &= -\frac{1}{8}-\frac{3}{32}\psi_2^2 \\ R_4 &= \frac{3}{16}+\frac{9}{32}\psi_2^2-\frac{1}{256}\psi_2^4. \end{aligned}$$

Proof. Consider the polynomial ring $\mathbb{Q}[x(t), y(t), z(t)]$ generated by the three functions

$$x(t) := \pi\theta(t)^4 \tag{7.22}$$

$$\begin{aligned} y(t) &:= -4\frac{\eta^{(1)}(t)}{\eta(t)} = \frac{\pi}{3}E_2(t) \\ &= \frac{\pi}{3}(1 - 24\sum_{1\le m}\frac{me^{-2\pi mt}}{1-e^{-2\pi mt}}). \end{aligned} \tag{7.23}$$

$$z(t) := y(t) + 4\frac{\theta^{(1)}(t)}{\theta(t)}. \tag{7.24}$$

These functions are all (essentially) modular forms of weight 2 on the theta group. (The Eisenstein series $E_2(t)$ is not quite a modular form.) One has

$$x(1) = \psi_2, \qquad y(1) = 1, \qquad z(1) = 0,$$

where the last two are derived using the transformation laws for $\theta(t)$ and $\eta(t)$. It follows that if $g(t) \in \mathbb{Q}[x(t), y(t), z(t)]$, then $g(1) \in \mathbb{Q}[\psi_2]$.

We claim that the polynomial ring $\mathbb{Q}[x(t), y(t), z(t)]$ is closed under differentiation $\frac{d}{dt}$. Indeed one has

$$\frac{d}{dt}x(t) = x(t)(z(t) - y(t)) \tag{7.25}$$

$$\frac{d}{dt}y(t) = -\frac{1}{2}y(t)^2 + \frac{1}{24}x(t)^2 + \frac{1}{8}z(t)^2 \tag{7.26}$$

$$\frac{d}{dt}z(t) = \frac{1}{6}x(t)^2 - y(t)z(t) - \frac{1}{2}z(t)^2. \tag{7.27}$$

These can be deduced using properties of derivatives of modular forms; the operator $\frac{d}{dt} + y(t)$ must take both $x(t)$ and $z(t)$ into a modular form of weight 4, the specific one being determined by the first few Fourier coefficients, and $\frac{d}{dt} + \frac{1}{2}y(t)$ does the same for $z(t)$. The procedure is that used in Lang [13, Chap. 10, Thm. 5.3, p. 161]. Furthermore, by symmetry, we observe that the subring $\mathbb{Q}[x(t)^2, y(t), z(t)]$ is also closed under differentiation.

Now, we observe that

$$\frac{d}{dt}\log\theta(t) = \frac{\frac{d}{dt}x(t)}{4x(t)} = \frac{1}{4}(z(t) - y(t)) \in \mathbb{Q}[x(t)^2, y(t), z(t)],$$

so for each $k \ge 1$, $\frac{d^k}{dt^k}\log\theta(t) \in \mathbb{Q}[x(t)^2, y(t), z(t)]$. The theorem follows upon evaluation at $t = 1$. The explicit formulas were found by computation. ■

We recall that the *cumulants* $\kappa_k = \kappa_k(P)$ of a probability distribution $P(dx)$ are defined in terms of the characteristic function of P by:

$$\kappa_k := i^{-k}\left(\frac{d^k}{dz^k}\log(\int_{-\infty}^{\infty} e^{izx}P(dx))\right)_{z=0}. \tag{7.28}$$

In particular, $\kappa_1(P)$ is the mean and $\kappa_2(P)$ is the variance of P. We extend this definition to the complex measures $\rho_{u,v}(x)dx$. The usefulness of cumulants for infinitely divisible distributions (as opposed to moments) is that they scale nicely with the parameter u.

Theorem 7.7. *For real $u > 0$ and real $|v| < u$, the mean value*

$$\kappa_1(\rho_{u,v}) = -\frac{iv}{2}.$$

For all $k \geq 2$, the k-th cumulant of the distribution $\rho_{u,v}(x)dx$ has the form

$$\kappa_k(\rho_{u,v}) = c_k u,$$

where c_k in $\mathbb{Q}[\psi_2]$, with

$$\psi_2 := \pi\theta(1)^4 = \frac{\pi^2}{\Gamma(3/4)^4}.$$

Moreover, if $k \geq 3$ is odd, then $c_k = 0$, while if $k \geq 2$ is even, then c_k is an even polynomial of degree k in ψ_2.

Remark. In particular, we have

$$\begin{aligned} c_2 &= -\frac{1}{2} + \frac{1}{8}\psi_2^2 = 4(R_1 + R_2) \\ c_4 &= -1 + \psi_2^2 + \frac{1}{16}\psi_2^4. \end{aligned}$$

Proof. We have:

$$\int_{-\infty}^{\infty} e^{izx}\rho_{u,v}(x)dx = \frac{1}{2\pi}\theta(1)^u \int_{-\infty}^{\infty} e^{izx} Z_{\mathbb{Q}}(-u, -\frac{u+v}{2} + ix)dx \tag{7.29}$$

$$= \frac{1}{2\pi}\theta(1)^u e^{(u+v)z/2} \int_{-\infty}^{\infty} e^{z(-\frac{u+v}{2}+ix)} Z_{\mathbb{Q}}(-u, -\frac{u+v}{2} + ix)dx. \tag{7.30}$$

Applying Lemma 4.8 with $Q(T) = 1$, $w = -u$, and $\sigma = -\frac{u+v}{2}$, replacing z by $-z$, and multiplying by $\theta(1)^u e^{(u+v)z/2}$, we obtain

$$\int_{-\infty}^{\infty} e^{izx}\rho_{u,v}(x)dx = \theta(1)^u e^{(u+v)z/2}\theta(e^{-2z})^{-u},$$

and thus

$$\log(\int_{-\infty}^{\infty} e^{izx}\rho_{u,v}(x)dx) = \frac{vz}{2} + u(\frac{z}{2} + \log\theta(1) - \log\theta(e^{-2z})).$$

We expand this in a Taylor expansion in z; clearly, v contributes only to the first cumulant. Conversely, since u is multiplied by an even function of z, it contributes only to the even cumulants; the claim for κ_1 follows immediately. Since

$$\log\theta(1+x) = \log\theta(1) + \sum_{k\geq 1} R_k x^k/k!,$$

we find that

$$\log\theta(1) - \log\theta(e^{-2z}) = -\sum_{k\geq 1} R_k(e^{-2z} - 1)^k/k!,$$

and thus has Taylor coefficients in $\mathbb{Q}[\psi_2]$. ■

Since the moments of a measure are polynomials in its cumulants, we obtain:

Corollary 7.8. *For real $u > 0$ and $|v| < u$ and integer $k \geq 0$, the moments*

$$M_k(u,v) = \int_{-\infty}^{\infty} x^k \rho_{u,v}(x)dx$$

satisfy $M_k(u,v) \in \mathbb{Q}[\psi_2][u,v]$, where $\psi_2 := \pi\theta(1)^4 = \frac{\pi^2}{\Gamma(3/4)^4}$.

We determine the mass of the Feller canonical measure.

Theorem 7.9. *For all $w \in \mathbb{C}$,*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \xi_{\mathbb{Q}}(w, \frac{w}{2} + ix)dx = \theta(1)^w(\frac{\psi_2^2}{16} - \frac{w}{8} - \frac{1}{4}).$$

In particular, the mass of the Feller canonical measure $M\{dx\}$is

$$\int_{-\infty}^{\infty} M(x)dx = \frac{\psi_2^2}{8} - \frac{1}{2}, \tag{7.31}$$

which is $\frac{\pi^2}{8}\theta(1)^8 - \frac{1}{2} \approx 1.8946$.

Proof. We deduce this using Lemma 4.8. From the uniform Schwartz property of $\xi_{\mathbb{Q}}$, we see that the left-hand side is entire in w. It thus suffices to prove the theorem when $\Re(w) < 0$. We then have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \xi_{\mathbb{Q}}(w, \frac{w}{2} + ix)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} Q(\frac{w}{2} + ix)Z_{\mathbb{Q}}(w, \frac{w}{2} + ix)dx \tag{7.32}$$

$$= Q(-\frac{d}{dz})(\theta(e^{-2z})^w)|_{z=0}, \tag{7.33}$$

where $Q(z) = z(z-w)/2w$. Differentiating and evaluating, we obtain:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \xi_{\mathbb{Q}}(w, \frac{w}{2} + ix)dx = \theta(1)^w\left((2R^2 + (w-2)R_1 + 2wR_1^2\right) = \theta(1)^w(\frac{\psi_2^2}{16} - \frac{w}{8} - \frac{1}{4}),$$

as required. Finally, since $M(x) = \frac{1}{\pi}\xi_{\mathbb{Q}}(0, -ix)$, we obtain the mass of the Feller canonical measure on taking $w = 0$. ∎

7.3. Location of Zeros

We know little about the location of the zeros of $\xi_{\mathbb{Q}}(-u, s)$, for negative real $-u$, aside from the general bound given by Lemma 4.2. We raise the following questions.

Question 1. Is $\Re(Z(w,s)) > 0$ in the entire real-codimension one cone $\mathcal{C}^- := \{(w,s) : w = u \in \mathbb{R},\ u < \Re(s) < 0\}$?

If true, this would extend the result of Theorem 7.1 to exclude zeros from the open cone. At $u = 0$ all the zeros are strictly outside the cone, so to prove this result it would suffice to show that there are never any zeros on the boundary of the cone.

Question 2. For each fixed $u < 0$ are the zeros of $\xi_{\mathbb{Q}}(u, s)$ confined to a vertical strip $|\Re(s)| < g(u)$ for some function $g(u)$?

The result of §5 shows that this is true on the boundary plane $u = 0$; perhaps it persists in the region $u < 0$. The results of §5 also suggest for $u > 0$ a limited movement of zeros in the vertical direction as u varies, so we ask the following question for negative u.

Question 3. For fixed $u < 0$ let $N_u(T)$ count (with multiplicities) the total number of zeros $\rho = \beta + i\gamma$ of $\xi_{\mathbb{Q}}(u, s)$ lying in the horizontal strip $|\Im(\rho)| \le T$. Is $N_u(T)$ finite for each $T > 0$, and if so, does it obey the same asymptotic formula as that for $u > 0$ in Theorem 6.2? Here we only ask for a remainder term smaller than the main term, possibly with a different dependence on u.

For real $-u < 0$, Theorem 7.1 shows that there are no zeros on the center line $\Re(s) = -\frac{u}{2}$. Thus the nonreal zeros always occur in quadruples $\{\rho, -\frac{u}{2} - \rho, \bar{\rho}, -\frac{u}{2} - \bar{\rho}\}$.

Question 4. The Riemann hypothesis is encoded in the location of the zeros of $\xi_{\mathbb{Q}}(0, s)$, asserting they are on the four lines $\Re(s) = \pm 1, \pm 2$. Indeed, the assertion that $\xi_{\mathbb{Q}}(0, s)$ has no zero with $|\Re(s)| < 1$ is equivalent to the Riemann hypothesis. Does the convolution semigroup structure for $u < 0$ play any factor in controlling the location of these zeros?

This question can be studied by considering more general convolution semigroups.

8. Location of Zeros: Complex w

The zero locus $\mathcal{Z}_{\mathbb{Q}}$ of the entire function $\xi_{\mathbb{Q}}(w, s)$ decomposes into a countable union of Riemann surfaces embedded in $\mathbb{C}^2$; we call these *components*. How many components are there in $\mathcal{Z}_{\mathbb{Q}}$? While we cannot answer this question, we can at least show that certain zeta zeros (on the slice $w = 1$) lie on the same component.

Lemma 8.1. *The Riemann zeta zeros $\rho_7 \approxeq \frac{1}{2} + 42.04i$ and $\rho_8 \approxeq \frac{1}{2} + 42.90i$ appearing as zeros of $Z_{\mathbb{Q}}(w, s)$ in the slice $w = 1$ belong to the same component of $\mathcal{Z}_{\mathbb{Q}}$.*

Proof. We deform $w = u$ through real values in the interval $1 \le u \le 2$. By numerical computation we find that at $u = \frac{3}{2}$ these two zeros have moved off the critical line to assume complex conjugate values.

We use the symmetry that when $w = u$ is real, if ρ is a zero, then so are $\bar{\rho}$, $1-\rho$ and $1-\bar{\rho}$. In particular, zeros cannot move off the critical line except by combining in pairs. As u changes, at some point u_0 they must coalesce on the critical line as a double zero, then as u changes go off the line, becoming a pair of complex conjugate zeros. The point of coalescence at u_0 of two zeros could be either the intersection of two different components of $\mathcal{Z}_{\mathbb{Q}}$ (the intersection having real codimension 4 in $\mathbb{C}^2$) or a single component of $\mathcal{Z}_{\mathbb{Q}}$ having a branch point of order two there (when viewed as projected on the w- plane.) The latter case must occur, because in the first case the movement of the zeros $\rho(u)$ would have a first derivative as a function of u which varies analytically in u at the critical point. This manifestly does not happen, because as a function of u the zeros first move vertically on the critical line, then change directions at u_0 to move horizontally off the line. Thus the component forms a single Riemann surface, with a path on it from $(u, s) = (1, \rho_7)$ to $(1, \rho_8)$. ■

It seems reasonable to guess that the zeta zeros $\{\rho = \sigma + it : \mathrm{Im}(t) > 0\}$ lie on the same component of $\mathcal{Z}_{\mathbb{Q}}$. If so, the same would hold for $\{\rho = \sigma + it : \mathrm{Im}(t) < 0\}$, since the zero set is invariant under complex conjugation, i.e. $\bar{\mathcal{Z}}_{\mathbb{Q}} = \mathcal{Z}_{\mathbb{Q}}$. The simplest hypothesis concerning the zero set would seem to be that it is the closure of a single irreducible complex-analytic variety of multiplicity one. However we do not have any strong evidence for this hypothesis.

9. General Number Fields

We now briefly consider the Arakelov zeta function for a general algebraic number field K. Many of the results extend to general K but the positivity property of Theorem 1.2 does not.

In the case of the Gaussian field $\mathbb{Q}(i)$, we have the identity

$$Z_{\mathbb{Q}(i)}(w, s) = 2Z_{\mathbb{Q}}(2w, 2s), \tag{9.1}$$

derived in the appendix. Thus all the results proved here immediately apply to $K = \mathbb{Q}(i)$.

For a general algebraic number field K, the Arakelov two-variable zeta function $Z_K(w, s)$ has a functional equation. Furthermore it can be shown that $\xi_K(w, s) := \frac{s(s-w)}{2w} Z_K(w, s)$ is an entire function of order one and infinite type of two variables, by generalization of the proofs for $K = \mathbb{Q}$.

The proofs given here for the distribution of zeros of $Z_{\mathbb{Q}}(w, s)$ for positive real w partially extend to general K. The proofs of confinement of the zeros to a vertial strip of width depending on w extend to a few fields such as the Gaussian field $\mathbb{Q}(i)$; they depend on the existence of an associated Dirichlet series with a nonempty half-plane of absolute convergence. For general K the Dirichlet series has a nonempty half-plane of convergence for w a positive integer, but perhaps not for any other values of w. One expects to get estimates for zeros to height T for such integer values of w . We do not know whether for fixed positive noninteger real w and general K the zeros are confined to a vertical strip of finite width, or that a generalization of the zero counting bounds in Theorem 6.2 holds, counting zeros in a horizontal strip $\Im(z) < T$. The latter seems plausible, because the zeros are confined at positive integer w, but if so, new proofs are needed.

The convolution semigroup property, of a family of complex-valued measures on a cone associated to negative real w, continues to hold for general number fields K. The associated measures are real-valued on the "critical line." However, the proof of positivity of such measures on the "critical line" for $\mathbb{Q}$ given in Theorem 1.2 extends only to a few specific number fields, such as $\mathbb{Q}(i)$. Our proof used a product formula for the associated modular form, which permits calculations with its logarithm and yields an explicit form for the associated Feller canonical measure. Such product forms exist only for modular forms all of whose zeros are at cusps. The modular forms associated to most imaginary quadratic fields generally do not have a product formula, because the associated modular forms have zeros in the interior of a fundamental domain, and the logarithm of such forms are multivalued functions.

There are imaginary quadratic number fields with class number one for which the positivity property does not hold. One can show that the positivity property holds for a field K if and only if $\xi_K(0, it)$ is nonnegative for all real t, where $\xi_K(w, s) = \frac{s(s-w)}{2w} Z_K(w, s)$. For $K = \mathbb{Q}(\sqrt{-2})$ one finds that

$$\xi_K(0, it) = 2\cos(t \log \sqrt{2})\xi_{\mathbb{Q}}(0, 2it),$$

which clearly has sign changes. We also found by computer calculation that $\xi_K(0, it)$ for $K = \mathbb{Q}(\sqrt{-11})$ changes sign between $t = 3.10$ and 3.15, and in addition on the line $w = -1$ there is a sign change, with $Z_K(-1, 4i) < 0$. For $K = \mathbb{Q}(\sqrt{-19})$ there is a sign change of $\xi_K(0, it)$ between $t = 2.0$ and $t = 2.1$. It remains conceivable that there exist imaginary quadratic fields having the positivity property, whose associated modular form does not have a product formula. In support of this, computer experiments for the imaginary quadratic number fields $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{-7})$ did not locate any sign changes for $\xi_K(0, it)$.

In a different direction, the positivity property of the convolution semigroup for negative real u on the "critical line" generalizes to certain classes of modular forms not associated to number fields, which do have a product formula, as we hope to treat elsewhere.

A. Appendix: Arakelov zeta function of van der Geer and Schoof

This appendix summarizes the framework of van der Geer and Schoof [11], and obtains explicit formulas for the two-variable zeta function for $K = \mathbb{Q}$ and $\mathbb{Q}(i)$. The expression of van der Geer and Schoof for the Arakelov two-variable zeta function is, formally,

$$\hat{\zeta}_K(w,s) \cong \int_{Pic(K)} e^{sh_0([D])+(w-s)h_1([D])} d[D] \ . \tag{A.1}$$

In this expression $h^0([D])$ resp. $h^1([D])$ are analogous to the "dimension" of a sheaf cohomology group. They give a direct definition of $h^0([D])$, and then define $h^1([D])$ indirectly [§] to be

$$h^1([D]) := h^0([\kappa_K] - [D]) \ , \tag{A.2}$$

where κ_K is the "canonical" Arakelov divisor for the ring of integers of K, which is what the Riemann-Roch formula predicts. We now define $Pic(K)$ and $h^0([D])$. The value $h^0([D])$ turns out to be the logarithm of a multivariable theta function at a specific point depending on $[D]$, see (A.16).

In what follows, let K be an algebraic number field, with O_K its ring of integers and Δ_K its discriminant. Set $[K:\mathbb{Q}] = n = r_1 + 2r_2$, with r_1 real places and r_2 complex places. We denote archimedean places of K by σ and non-archimedean places by ν.

Definition A.1. (i) An *Arakelov divisor* D is a formal finite sum over the non-archimedean places ν of K and the $r_1 + r_2$ archimedean places σ,

$$D = \sum_{\nu<\infty} n_\nu \nu \quad + \sum_{\sigma} x_\sigma \sigma$$

in which each n_ν is an integer and each x_ν is a real number (even at a complex place σ.)

(ii) An Arakelov divisor D is *principal* if there is an element $\alpha \in K^*$ such that

$$D = (\alpha) = \sum_{\nu<\infty} ord_\nu(\alpha)\nu + \sum_{\sigma} x_\sigma(\alpha)\sigma,$$

in which $x_\sigma(\alpha)$ equals $\log|\sigma(\alpha)|$ or $2\log|\sigma(\alpha)|$ according as σ is a real place or a complex place. Here $\sigma(.)$ runs over all embeddings of K into $\mathbb{C}$, with the convention that only one out of each conjugate complex pair of complex embeddings is used.

(iii) $Div(K)$ denotes the group of Arakelov divisors (under addition). The *Arakelov divisor class group* $Pic(K)$ is the quotient group by the subgroup of principal Arakelov divisors. The divisor class of D is denoted $[D]$.

The roots of unity μ_K in K have Arakelov divisor zero. They fit in the exact sequence

$$0 \to \mu_K \to K^* \to Div(K) \to Pic(K) \to 0. \tag{A.3}$$

Definition A.2. The *degree* $deg(D)$ of an Arakelov divisor is the real number

$$deg(D) := \sum_{\nu<\infty} n_\nu \log N\nu + \sum_{\sigma} x_\sigma.$$

Here $N\nu := |O_K/P_\nu|$, where $P_\nu = \{\alpha \in O_K : |\alpha|_\nu < 1\}$, in which $N\nu$ is the number of elements in the residue field of ν.

[§] A. Borisov [6] has given a direct definition of $h^1([D])$ in some cases, with a proof of the formula (A.2).

Principal divisors have degree zero, so the degree deg([D]) is well-defined on Arakelov divisor classes.

Definition A.3. The *norm* $N(D)$ of an Arakelov divisor D is

$$N(D) = \exp(deg(D)) = \prod_{\nu<\infty} (N\nu)^{n_\nu} \prod_{\sigma} e^{x_\sigma}.$$

Definition A.4. (i) The *ideal* I_D associated to an Arakelov divisor D at the finite places is the fractional ideal

$$I_D := \prod_{\nu<\infty} P_\nu^{-n_\nu},$$

where P_ν denotes the prime ideal at ν.

(ii) The lattice structure associated to an Arakelov divisor D at the infinite places is a positive inner product on $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$. defined as follows. At a real place, x_σ determines a scalar product on $\mathbb{R}$ such that $||1||^2 = e^{-2x_\sigma}$. At a complex place x_σ determines a Hermitian inner product on $\mathbb{C}$ such that $||1||^2 = 2e^{-x_\sigma}$. The combined inner product is

$$||(z_\sigma)||_D^2 := \sum_\sigma |z_\sigma|^2 ||1||_\sigma^2.$$

The *(metrized) lattice* Λ_D associated to D is the fractional ideal I_D (viewed as a subset of K) embedded in $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ as Galois conjugates of each element α, with distance function measured by this inner product.

The number field K embeds as a dense subset of $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, while each fractional ideal I_D embeds discretely as a lattice in this space. The archimedean coordinates x_σ define a metric structure at the infinite places such that

$$Covol(\Lambda_D) = \frac{\sqrt{\Delta_K}}{N(D)},$$

where Δ_K is the discriminant of K. The Arakelov class group $Pic(K)$ parametrizes isometry classes of lattices that have compatible O_K-structures under multiplication. Following Szpiro, the *Euler-Poincaré characteristic* $\chi(D)$ of an Arakelov divisor D is defined as

$$\chi(D) := -\log(covol(\Lambda_D)) = deg(D) - \frac{1}{2}\log \Delta_K.$$

It is well-defined on divisor classes $[D]$. In general the Arakelov class group

$$Pic(K) \simeq Cl(K) \times \mathbb{R} \times \mathbf{T}^{r_1+r_2-1}, \tag{A.4}$$

where $Cl(K)$ denotes the (wide) ideal class group of K, and the second factors combined are $\mathbb{R}^{r_1+r_2}/U(K)$ where $U(K)$ is an r_1+r_2-1 dimensional lattice given by logarithms of (absolute values of) Galois conjugates of units. Note that $Pic^0(K)$, the group of divisor classes of degree 0, is compact, and its volume is $h_K R_K$ where h_K and R_K are the class number and regulator of K, respectively.

Definition A.5. The *canonical divisor* κ_K of a number field K is the Arakelov divisor whose associated ideal part I_{κ_K} is the inverse different $\mathfrak{d}_K^{-1}$ for $K/\mathbb{Q}$, and all of whose archimedean coordinates $x_\sigma = 0$.

These definitions imply that

$$N(\kappa_K) = N(I_{\kappa_K})^{-1} = N(\mathfrak{d}_K) = \Delta_K.$$

Definition A.6. (i) An Arakelov divisor D is *effective* if $O_K \subseteq I_D$.

(ii) The *effectivity* $e(D)$ of an effective divisor D is

$$e(D) = \exp(-\pi ||1||_D^2),$$

in which

$$||1||_D^2 := \sum_{\sigma \quad real} e^{-2x_\sigma} + \sum_{\sigma \quad complex} 2e^{-x_\sigma}.$$

The *effectivity* $e(D)$ of a non-effective divisor is 0.

The effective divisors are those Arakelov divisors with each $n_\nu \geq 0$ and the effectivity $e(D)$ of any divisor takes a value $0 \leq e(D) < 1$. The only "functions" $\alpha \in K^*$ whose associated principal divisors (α) are effective are the roots of unity in K, whose associated Arakelov divisor is 0, the identity element. By convention we add a symbol (0) to represent a "divisor at infinity" corresponding to the element $0 \in K$, with the convention that $(0) + D = (0)$ for all Arakelov divisors $[D]$ and we define the effectivity $e((0)) := 1$.

Definition A.7. (i) The *order* $H^0(D)$ of the group of effective divisors associated to an Arakelov divisor D is

$$H^0(D) := \sum_{\alpha \in I_D} e((\alpha) + D).$$

This sum includes a term $\alpha = 0 \in K$, with the convention that $e((0) + D)) := 1$, so that $H^0([D]) \geq 1$.

(ii) The *effectivity dimension* $h^0(D)$ of $H^0(D)$ is given by

$$h^0(D) = \log H^0(D).$$

One has $h^0(D) \geq 0$.

The quantities $H^0(D)$ and $h^0(D)$ are constant for all divisors in an Arakelov divisor class $[D] \in Pic(K)$ and may therefore be denoted $H^0([D])$ and $h^0([D])$, respectively. van der Geer and Schoof [11, Prop. 1] state the following result.

Theorem A.1. **(Riemann-Roch Theorem for Number Fields)** *For any algebraic number field K and any Arakelov divisor class $[D] \in Pic(K)$,*

$$h^0([D]) - h^0([\kappa_K] - [D]) = \deg([D]) - \frac{1}{2}\deg([\kappa_K]), \tag{A.5}$$

in which κ_K is the canonical Arakelov divisor for K, and $\deg([\kappa_K]) = \log \Delta_K$.

van der Geer and Schoof defined a new invariant $\eta(K)$ of a number field K, and an Arakelov analogue of the genus $g(K)$ of K.

Definition A.8. (i) The invariant $\eta(K)$ of K is defined by

$$\eta(K) := H^0(O_K). \tag{A.6}$$

(ii) The *genus* $g(K)$ of K is defined by

$$g(K) := h^0(\kappa_K) = h^0(O_K) + \frac{1}{2}\log \Delta_K = \log(\eta(K)\sqrt{\Delta_K}). \tag{A.7}$$

For the rational number field

$$\eta(\mathbb{Q}) = \omega := \frac{\pi^{1/4}}{\Gamma(\frac{3}{4})} \approx 1.08643, \tag{A.8}$$

and $g(\mathbb{Q}) = \log \eta(\mathbb{Q}) \approx 0.0829015$. The value $\omega = \theta(1)$, where $\theta(\cdot)$ is the theta function (1.2). One also has

$$\eta(\mathbb{Q}(i)) = (\frac{2+\sqrt{2}}{4})\omega^2,$$

see [11, pp. 16-17].

The genus of a function field is usually defined to be the dimension $h^0(\kappa)$ of the vector space of effective divisors for the canonical class κ; this motivates the definition of the genus $g(K)$ of a number field. For a function field the degree $deg(\kappa)$ of the canonical class is $2g-2$, so one may consider

$$\tilde{g}(K) := \frac{1}{2}(\deg(\kappa_K) + 2)$$

as a second analogue of genus for a number field. This analogue appears on the right side of the Riemann-Roch theorem for number fields. One has

$$\tilde{g}(K) = 1 + \frac{1}{2}\log \Delta_K. \tag{A.9}$$

In particular $\tilde{g}(\mathbb{Q}) = 1$, and $\tilde{g}(\mathbb{Q}(i)) = 1 + 2\log 2$. The two notions of genus agree in the function field case and differ in the number field case.

Below we obtain explicit integral formulas for the two-variable zeta functions for $K = \mathbb{Q}$ and $\mathbb{Q}(i)$, of the form (A.1), and also indicate the form of the two-variable zeta function for a general algebraic number field K.

To define an integral over the Arakelov class group, one must specify a measure on the group. For compact groups it is Haar measure, and on noncompact additive groups $\mathbb{R}$ it is dx. It is convenient to replace additive groups by multiplicative group $\mathbb{R}_{>0}$ using the change of variable $y_\sigma = e^{-x_\sigma}$ at real places and the appropriate measure becomes $\frac{dy}{y}$. For $\mathbb{Q}$ and $\mathbb{Q}(i)$ the Arakelov class group is isomorphic to $\mathbb{R}$.

Case $K = \mathbb{Q}$. There is a single real place σ. For representatives of Arakelov divisor classes $[D]$ we may take D to have ideal $O_{\mathbb{Q}} = \mathbb{Z}$ and with value at the infinite place $x_\sigma \in \mathbb{R}$ arbitrary, with measure dx at the infinite place. Thus the Arakelov class group $Pic(\mathbb{Q}) \equiv \mathbb{R}$, the additive group, with $x \in \mathbb{R}$ being the degree of the divisor. We have

$$\begin{aligned} H^0([D]) &= \sum_{n\in\mathbb{Z}} e((n) + D) \\ &= 1 + \sum_{n\in\mathbb{Z}\setminus\{0\}} \exp(-\pi e^{2\log|n| - 2x_\sigma}) \\ &= \sum_{n\in\mathbb{Z}} e^{-\pi n^2 y_\sigma^2}, \end{aligned} \tag{A.10}$$

in which one sets $y_\sigma = e^{-x_\sigma}$. Using the multiplicative change of variable $y_\sigma = e^{-x_\sigma}$ we identify $Pic(\mathbb{Q})$ with the multiplicative group $\equiv \mathbb{R}_{>0}$ with measure $\frac{dy}{y}$. Thus we have

$$H^0([D]) = \theta(y_\sigma^2),$$

in terms of the theta function (1.2).

The different $\mathfrak{d}_{\mathbb{Q}} = (1)$, so the canonical divisor $\kappa_{\mathbb{Q}} = 0$. Consequently we have

$$H^1([D]) := H^0(-[D]) = \theta(\frac{1}{y_\sigma^2}).$$

We obtain

$$\begin{aligned} Z_{\mathbb{Q}}(w,s) &:= \int_{Pic(\mathbb{Q})} e^{sh_0([D])+(w-s)h_1([D])} d[D] \\ &= \int_{Pic(\mathbb{Q})} H^0([D])^s H^1([D])^{w-s} d[D] \\ &= \int_0^\infty \theta(y^2)^s \theta(\frac{1}{y^2})^{w-s} \frac{dy}{y}. \end{aligned} \tag{A.11}$$

Case $K = \mathbb{Q}(i)$. There is a single complex place σ. The class number of $O_K = \mathbb{Z}[i]$ is one, so all Arakelov divisor classes $[D]$ in $Pic(\mathbb{Q}(i))$ have a representative $D = x_\sigma \sigma$, whose associated ideal is O_K. Thus $Pic(K) \equiv \mathbb{R}$ as an additive group. Letting $\alpha = m + ni \in \mathbb{Z}[i]$, we have

$$\begin{aligned} H^0([D]) &= \sum_{\alpha \in \mathbb{Z}[i]} e((\alpha) + D) = 1 + \sum_{\alpha \in \mathbb{Z}[i] \setminus \{0\}} \exp(-2\pi e^{2\log|\alpha| - x_\sigma}) \\ &= \sum_{m\in\mathbb{Z}} \sum_{n \in \mathbb{Z}} e^{-2\pi(m^2+n^2)y_\sigma}, \\ &= \left(\sum_{m \in \mathbb{Z}} e^{-2\pi m^2 y_\sigma} \right)^2. \end{aligned} \tag{A.12}$$

in which $y_\sigma = e^{-x_\sigma}$. Thus

$$H^0([D]) = \theta(2y_\sigma)^2. \tag{A.13}$$

The different $\mathfrak{d}_{\mathbb{Q}(i)} = ((1+i)^2) = (2)$, and the canonical divisor $\kappa_{\mathbb{Q}(i)} = \nu_{(1+i)}^{-2} = (\frac{1}{2})$. Consequently we have

$$H^1([D]) := H^0(\kappa_{\mathbb{Q}(i)} - [D]) = \theta(\frac{1}{2y_\sigma}).$$

We obtain

$$\begin{aligned} Z_{\mathbb{Q}(i)}(w,s) &:= \int_{Pic(\mathbb{Q}(i))} e^{sh_0([D])+(w-s)h_1([D])} d[D] \\ &= \int_{Pic(\mathbb{Q}(i))} H^0([D])^s H^1([D])^{w-s} d[D] \\ &= \int_0^\infty \theta(2y)^{2s} \theta(\frac{1}{2y})^{2w-2s} \frac{dy}{y} \\ &= 2\int_0^\infty \theta(t^2)^{2s} \theta(\frac{1}{t^2})^{2w-2s} \frac{dt}{t} \end{aligned} \tag{A.14}$$

using the change of variables $t^2 = 2y$. Comparing this with (A.11) yields

$$Z_{\mathbb{Q}(i)}(w,s) = 2Z_{\mathbb{Q}}(2w,2s). \tag{A.15}$$

General Number Fields. Let K be an algebraic number field, of degree $[K:\mathbb{Q}] = n$. We follow Lang [14, Chapter 13] for Hecke's functional equation for the Dedekind zeta function. One can show that

$$Z_K(w,s) = \frac{2}{w(K)} \sum_{\mathfrak{a}\in Cl(K)} \int_0^\infty \left(\int_E \theta(t^{2/n}\mathfrak{c},\mathfrak{a})^s \theta(t^{-2/n}\mathfrak{c}^{-1}, \mathfrak{d}_K \mathfrak{a}^{-1})^{w-s} d^*\mathfrak{c} \right) \frac{dt}{t}, \tag{A.16}$$

which uses a decomposition of the Arakelov class group (A.4). Here $\mathfrak{a}$ runs over a set of representatives of the (wide) ideal class group, $w(K)$ counts the number of roots of unity in K, and E is a fundamental domain in the (logarithmic) space of units, with Haar measure $d^*\mathfrak{c}$. The theta function $\theta(t^{2/n}\mathfrak{c},\mathfrak{a})$ is defined in Lang [14, p.253] and satisfies the functional equation

$$\theta(t^{2/n}\mathfrak{c},\mathfrak{a}) = \frac{1}{t}\theta(t^{-2/n}\mathfrak{c}^{-1}, \mathfrak{d}_K\mathfrak{a}^{-1}), \tag{A.17}$$

using the fact that $||\mathfrak{c}|| = 1$, see Lang [14, p.257]. Using this functional equation and the substitution $u = t^{-2}$ one obtains

$$Z_K(w,s) = \frac{1}{w(K)} \sum_{\mathfrak{a}\in Cl(K)} \int_0^\infty \left(\int_E \theta(u^{1/n}\mathfrak{c}^{-1}, \mathfrak{d}_K\mathfrak{a}^{-1})^w d^*\mathfrak{c} \right) u^{s/2} \frac{du}{u}. \tag{A.18}$$

One has the functional equation

$$Z_K(w,s) = Z_K(w, w-s).$$

For $w = 1$ one recovers the completed Dedekind zeta function

$$Z_K(1,s) = \hat{\zeta}_K(s) := A_K^s \Gamma(\frac{s}{2})^{r_1} \Gamma(s)^{r_2} \zeta_K(s), \tag{A.19}$$

in which $A_K := 2^{-r_2}\pi^{-n/2}\Delta_K^{1/2}$, see Lang [14, p. 254].

References

[1] G. E. Andrews, *The Theory of Partitions*, Addison-Wesley: Reading, Mass. 1976. (Reprint: Cambridge University Press 1998.)

[2] G. E. Andrews, R. Askey and R. Roy, *Special Functions,* Cambridge Univ. Press, Cambridge 1999.

[3] T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory,* Springer: New York 1976.

[4] P. Biane, J. Pitman and M. Yor, Probability laws related to the Jacobi theta and Riemann zeta functions, and Brownian excursions, Bull. Amer. Math. Soc. **38** (2001), 435–465.

[5] E. Bombieri and D. A. Hejhal, On the distribution of zeros of linear combinations of Euler products, Duke Math. J. **80** (1995), 821–862.

[6] A. Borisov, Convolution structures and arithmetic cohomology, eprint: `arXiv: math.AG980751 v3 3 Jan 2001`.

[7] R. W. Bruggeman, *Families of Automorphic Forms*, Birkhäuser Verlag: Basel 1994.

[8] J. B. Conrey and A. Ghosh, Turán inequalitites and zeros of Dirichlet series associated with certain cusp forms, Trans. Amer. Math. Soc. **342** (1994), 407–419.

[9] H. Davenport, *Multiplicative Number Theory, Second edition,* Revised by H. Montgomery, Springer-Verlag: New York 1980.

[10] W. Feller, *An Introduction to Probability Theory and its Applications, Volume II*, Second Edition, John Wiley & Sons: New York 1971.

[11] G. van der Geer and R. Schoof, Effectivity of Arakelov Divisors and the theta divisor of a number field, Selecta Math., New Series **6** (2000), 377–398. eprint: `arXiv math.AG/9802121`.

[12] D. A. Hejhal, On a result of Selberg concerning zeros of linear combinations of L-functions, Internat. Mat. Research Notices 2000, No. 11, 551–577.

[13] S. Lang, *Introduction to Modular Forms*, Springer-Verlag: New York 1976.

[14] S. Lang, *Algebraic Number Theory. Second Edition*, Springer-Verlag: New York 1994.

[15] J. Lehner, The Fourier coefficients of automorphic forms on horocyclic groups II, Michigan Math. J. **6** (1959), 173–193.

[16] J. Lehner, Magnitude of the Fourier coefficients of automorphic forms of negative dimension, Bull. Amer. Math. Soc. **67** (1961), 603–606.

[17] J. Lehner, *Discontinuous Groups and Arithmetic Subgroups,* Mathematical Surveys, Number VIII, Amer. Math. Soc.: Providence, RI 1964.

[18] R. Pellikaan, On special divisors and the two variable zeta function of algebraic curves over finite fields, in: *Arithmetic, Geometry and Coding Theory,* (R. Pellikaan, M. Perret and S. G. Vladut, Eds.), Walter de Gruyter: Berlin 1996, pp. 175–184.

[19] H. Petersson, Über automorphe Orthogonalfunktionen und die Konstruktion der automorphen Formen von positiver reeller Dimension, Math. Ann. **127** (1954), 33–81.

[20] H. Petersson, Über Betragmittelwerte und die Fourier-Koeffizienten der ganzen automorphen Formen, Arch. Math. (Basel), **9** (1958), 176–182.

[21] J. Pitman and M. Yor, Infinitely divisible laws associated to hyperbolic functions, Univ. Calif.-Berkeley Stat. Technical Rept. No. 581, revised Feb. 2001.

[22] H. Rademacher, On the expansion of the partition function in a series, Ann. Math. **44** (1943), 416–422.

[23] L. I. Ronkin, *Introduction to the Theory of Entire Functions of Several Variables,* Amer. Math. Soc.: Providence, RI 1974.

[24] L. I. Ronkin, Entire Functions, in: (G. M. Khenkin, Ed.) *Several Complex Variables III*, Springer-Verlag: New York 1989, pp. 1–30. [Volume 9 of Encyclopedia of Mathematical Sciences].

[25] W. Stoll, *Holomorphic Functions of Finite Order in Several Complex Variables*, CBMS Publication No. 21, Amer. Math. Soc.: Providence, RI 1974.

email:
jcl@research.att.com
rains@research.att.com